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TRANSFORMATION THEORY FOR LEBESGUE INTEGRALS

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CHAPTER I

INTRODUCTION

In the theory of functions defined on subsets of q-dimensional Euclidean space, it is often necessary to make use of the theory and application of transformations for the Lebesgue integral. It is the purpose of this study to develop a transformation theory which lends itself to the proof of a wide variety of change of variable theorems for the Lebesgue integral, and which extends beyond, but includes, the elementary transformation and change of variable theory of the literature. It is assumed that the reader is acquainted with certain areas of real analysis, including elementary topology, measure theory, and Lebesgue integration.

In Chapter II a number of approximation theorems are proved. In addition to their importance in approximation theory, these theorems form the nucleus of the proofs of the change of variable theorems in the real line. Furthermore, Chapter II contains a brief discussion of the concepts of upper and lower semicontinuity, notions essential to the understanding of the theorems presented in this and in subsequent chapters.

In Chapter III two transformation theorems are proved which describe the effect of a change of variable in a Lebesgue integral in the real line. The first theorem is concerned with a change of variable in the integral of a bounded Lebesgue measurable function defined on a closed interval. The second theorem is similar, except that now the function is assumed to be Lebesgue summable over the interval.

In a change of variable theory the transition from the one dimensional case to higher dimensional cases is extremely difficult. In higher dimensions the nature of the problem changes to the extent that additional concepts must be defined and developed. These concepts, upon which a change of variable theory can be based, are the subjects of Chapters IV and V. By using the theory of Lebesgue measure, one can prove various general change of variable theorems without making reference to the ordinary Jacobian, or even to differentiability. It is in this connection that the concept of the generalized Jacobian is introduced. As anticipated, the generalized Jacobian plays the rôle of the ordinary Jacobian in the transformation theorems which are considered in Chapter VI.

The theory of differentiation of the Lebesgue integral in q-dimensional Euclidean space $(q > 1)$ is more involved than the corresponding theory in the one-dimensional case. For this reason the concept of the regular derivative is introduced; and it is then used to prove Lebesgue's famous theorem on differentiating the indefinite integral.

In Chapter VI two important transformation formulas for the integral are proved by making use of the differentiation theory of Chapter V. The proofs given here possess the outstanding feature that the transformation is not required to be differentiable in its domain. Although this non-inductive method of proof is dependent upon all of the previous theory, the proofs are much simpler in principle than those involved in the inductive approaches.

In the final two chapters of the text, various special classes of transformations are considered. In applications the mappings involved are usually "well behaved" so that their differentiability properties may be utilized. In this case the ordinary Jacobian will exist and assume its rôle as the local magnification element in the transformation formula. A large part of Chapter VII is devoted to showing the equivalence of the ordinary and the generalized Jacobian whenever the former exists. In the final chapter four special change of variable theorems are considered; it is shown that they are particular cases of the general transformation theorem presented in Chapter VI. An example illustrating a particular change of variable problem is discussed at the end of this chapter.

CHAPTER II

THE THEOREM OF VITALI-CARATHÉODORY AND ADDITIONAL APPROXIMATION LEMMAS

In this chapter the concepts of "upper semicontinuity" and "lower semicontinuity" will be defined and used to prove certain important approximation theorems for Lebesgue measurable functions. It is assumed here that the reader has a certain familiarity with these concepts, although certain general facts and theorems regarding them are given below. Other common straightforward theorems are given without proof in the Appendix, and references to these theorems are made when appropriate. The abbreviations U.S.C. and L.S.C. will be used for upper semicontinuous and lower semicontinuous respectively.

<u>Definition 2.1</u>: An extended-real-valued function f with domain $D \subset R_q$ (R_q is q-dimensional Euclidean Space) is said to be $\left\{\begin{matrix} \text{U.S.C.} \\ \text{L.S.C.} \end{matrix}\right\}$ on D if and only if for every $\overline{a} \in D$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that $\left\{\begin{matrix} f(\overline{x}) - f(\overline{a}) < \epsilon \\ -\epsilon < f(\overline{x}) - f(\overline{a}) \end{matrix}\right\}$ for every $\overline{x} \in N(\overline{a}:\delta) \cap D$ (the notation $\overline{x}$ is used to denote the q-tuple $(x_1, x_2, \ldots, x_q)$ in R_q).

Remark: The concepts of U.S.C. and L.S.C. may also be defined in the following manner. Let $\sup f(\overline{a})$ and $\inf f(\overline{a})$ be defined at the point $\overline{a} \in D$ by:

$$\sup f(\overline{a}) = \sup \left\{ f(\overline{x}) \mid \overline{x} \in N(\overline{a}:\delta) \cap D \right\}$$

and

$$\inf f(\bar{a}) = \inf \left\{ f(\bar{x}) \mid \bar{x} \in N(\bar{a}:\delta) \cap D \right\}.$$

As δ tends to zero, the two numbers $\sup f(\bar{a})$ and $\inf f(\bar{a})$ tend monotonically towards limits (finite or infinite). For that reason the limits $\overline{\lim}\, f(\bar{a})$ and $\underline{\lim}\, f(\bar{a})$ defined by

$$\overline{\lim}\, f(\bar{a}) = \lim_{\delta \to 0^+} \sup f(\bar{a})$$

and

$$\underline{\lim}\, f(\bar{a}) = \lim_{\delta \to 0^+} \inf f(\bar{a})$$

always exist. It is evident that

$$\underline{\lim}\, f(\bar{a}) \leq f(\bar{a}) \leq \overline{\lim}\, f(\bar{a})$$

for each fixed $\bar{a} \in D$. Hence the extended-real-valued function f defined as $D \subset R_q$ is said to be $\left\{ \begin{matrix} \text{U.S.C.} \\ \text{L.S.C.} \end{matrix} \right\}$ on D if and only if $\left\{ \begin{matrix} \overline{\lim}\, f(\bar{a}) = f(\bar{a}) \\ \underline{\lim}\, f(\bar{a}) = f(\bar{a}) \end{matrix} \right\}$. Throughout the text this definition and Definition 2.1 will be used interchangeably since their equivalence is obvious.

<u>Lemma 2.2</u>: Let G be an open set in R_q and let K_G be the characteristic function of G. Then K_G is L.S.C. on R_q.

<u>Proof</u>: Suppose $\bar{a}$ is a point in G. Since G is open, it is evident that $\underline{\lim}\, K_{G(\bar{a})} = K_{G(\bar{a})} = 1$. Thus K_G is L.S.C. on G. Now suppose $\bar{a} \in R_q - G$. Then it is equally obvious that

$$\lim_{\delta \to 0^+} \inf \left\{ K_{G(\bar{x})} \mid \bar{x} \in N(\bar{a}:\delta) \right\} = K_{G(\bar{a})} = 0 \;,$$

so K_G is L.S.C. on $R_q - G$. Thus the proof of the lemma is complete. ∎

Lemma 2.3: Let Γ be a nonempty family of extended-real-valued functions with domain D in R_q. Let $\overline{x}_o$ be a point in D at which all the functions $f \in \Gamma$ are L.S.C. Then the function g defined by $g(\overline{x}) = \sup\left\{ f(\overline{x}) \mid f \in \Gamma \right\}$, for each $\overline{x} \in D$, is also L.S.C. at $\overline{x}_o$.

Proof: Let $f \in \Gamma$. Then $f(\overline{x}) \leq g(\overline{x})$ for every $\overline{x} \in D$. Hence

$$\underline{\lim}\, g(\overline{x}_o) \geq \underline{\lim}\, f(\overline{x}_o) = f(\overline{x}_o)$$

for each $f \in \Gamma$ since each f is L.S.C. at $\overline{x}_o$. Thus it follows that $\underline{\lim}\, g(\overline{x}_o) = g(\overline{x}_o)$ since $\underline{\lim}\, g(\overline{x}_o) \leq g(\overline{x}_o)$ for any extended-real-valued function g defined on D. Thus g is L.S.C. at $\overline{x}_o$, and the theorem is proved. ∎

Lemma 2.4: Let $f : R_q \to R^*$ be a nonnegative L.S.C. function. Then it follows that the reciprocal function $\frac{1}{f}$ is U.S.C. on R_q.

Remark: It is understood in the lemma that $\frac{1}{f(\overline{x})}$ is replaced by 0 when $f(\overline{x}) = +\infty$ and $\frac{1}{f(\overline{x})}$ is replaced by $+\infty$ when $f(\overline{x}) = 0$.

Proof: The proof of this lemma is a straightforward consequence of Definition 2.1 and will not be given here.

Lemma 2.5: (Approximation Lemma): Let $f : R_q \to R^*$ where f is nonnegative and Lebesgue measurable. Then for every $\epsilon > 0$ there exists a function $g : R_q \to R^*$ having the following properties:

(i) g is lower semicontinuous.

(ii) $g(\overline{x}) \leq f(\overline{x})$ for every $\overline{x} \in R_q$.

(iii) $\int_{R_q} (g-f)\, d\mu < \epsilon.$

Note: The condition (iii) requires the following clarification. If $f(\overline{x}) = g(\overline{x}) = +\infty$, $g(\overline{x}) - f(\overline{x})$ will not be defined; hence, define $h : R_q \to R^*$ by

$$\begin{aligned} h(\overline{x}) &= g(\overline{x}) - f(x), \quad \text{unless} \quad g(\overline{x}) = f(\overline{x}) = +\infty ; \\ &= 0 \quad , \text{ whenever } \quad g(\overline{x}) = f(\overline{x}) = +\infty . \end{aligned}$$

Thus Equation (iii) will mean

$$\int_{R_q} h\, d\mu < \epsilon .$$

<u>Proof</u>: Case 1. Let f be bounded and have bounded support. A bounded support of f is defined to be a bounded Lebesgue measurable set A such that $f(\overline{x}) = 0$ for $\overline{x} \notin A$. In particular, let A be a bounded support of f. Define, for $j = 1,2,\ldots,$ the set A_j by

$$A_j = \left\{ \overline{x} \mid \overline{x} \in A \text{ and } (j-1)\delta \leq f(\overline{x}) < j\delta \right\}$$

for some fixed positive number δ. Since A_j is Lebesgue measurable for each $j = 1,2,\ldots,$ by a standard property of Lebesgue measure, there corresponds an open set $G_j \supset A_j$ such that $\mu(G_j - A_j) < \frac{1}{j2^j}$. Define $g : R_q \to R$ by

$$g(\overline{x}) = \sum_{j=1}^{\infty} j\, \delta\, K_{G_j}(\overline{x}) ,$$

where K_{G_j} is the characteristic function of G_j. It will now be shown that g satisfies the condition of the theorem; this will complete the proof in case 1.

(i) First, Lemma 2.2 implies that K_{G_j} is L.S.C. on $R_q (j=1,2,\dots)$; thus, the function

$$g(\overline{x}) = \sum_{j=1}^{\infty} j\,\delta\,K_{G_j}(\overline{x})$$

is the supremum of a nonempty family of L.S.C. functions and is itself L.S.C. by Lemma 2.3.

(ii) Also $g(\overline{x}) \geq f(\overline{x})$ for each $\overline{x} \in R_q$. If $\overline{x} \in A_j$ for some j, then $\overline{x} \in G_j$, so that $f(\overline{x}) < j\,\delta$ and $g(\overline{x}) \geq j\,\delta > f(\overline{x})$. If $\overline{x} \notin A = \bigcup_{j=1}^{\infty} A_j$, then $f(\overline{x}) = 0$ so that $g(\overline{x}) \geq f(\overline{x})$ in this case also.

(iii) It remains to verify that

$$\int_{R_q} (g-f)\,d\mu < \epsilon \ .$$

In particular, let $\epsilon > 0$ be given. Then let

$$\delta = \frac{\epsilon}{1 + \mu(A)}$$

in the definition of A_j. Hence

$$\int_A (g(\overline{x}) - f(\overline{x}))\,d\mu = \int_A \left[\sum_{j=1}^{\infty} j\,\delta\,K_{G_j}(\overline{x}) - f(\overline{x}) \right] d\mu =$$

$$= \int_A \sum_{j=1}^{\infty} j\,\delta\,K_{G_j}(\overline{x})\,d\mu - \int_A f\,d\mu =$$

$$= \sum_{j=1}^{\infty} \int_A j\,\delta\,K_{G_j}(\overline{x})\,d\mu - \int_{\bigcup_{j=1}^{\infty} A_j} f\,d\mu \leq \sum_{j=1}^{\infty} j\,\delta\,\mu(G_j) - \sum_{j=1}^{\infty} (j-1)\,\delta\,\mu(A_j) =$$

$$= \sum_{j=1}^{\infty} j\,\delta(\mu(G_j - A_j)) + \sum_{j=1}^{\infty} \delta\,\mu(A_j) = \delta \sum_{j=1}^{\infty} \frac{j}{j2^j} + \delta\,\mu(A) = \epsilon .$$

This completes the proof of case 1 of Lemma 2.5.

Case 2. In the general case define a sequence $h_n \uparrow f$ as follows. For each $n = 0,1,2,\ldots$, h_n is defined on R_q by:

$$\begin{aligned} h_n(\overline{x}) &= f(\overline{x}) \ , \quad \text{if } \rho(\overline{x},\overline{0}) < n \text{ and } f(\overline{x}) \leq n \ ; \\ &= n \ , \quad \text{if } \rho(\overline{x},\overline{0}) < n \text{ and } f(\overline{x}) > n \ ; \\ &= 0 \ , \quad \text{if } \rho(\overline{x},\overline{0}) \geq n \ . \end{aligned}$$

Now define $f_n(\overline{x}) = h_n(\overline{x}) - h_{n-1}(\overline{x})$ for each $n = 1,2,\ldots$ and for each $\overline{x} \in R_q$. It follows that

$$f(\overline{x}) = \sum_{n=1}^{\infty} f_n(\overline{x})$$

since

$$\sum_{n=1}^{\infty} f_n(\overline{x}) = \lim_{n\to\infty} \sum_{k=1}^{n} \left[h_k(\overline{x}) - h_{k-1}(\overline{x}) \right] = \lim_{h\to\infty} h_n(\overline{x}).$$

If $f(\overline{x}) = +\infty$, $h_n(\overline{x}) \to +\infty$. If $f(\overline{x}) < +\infty$, $h_n(x) \to f(\overline{x})$ by definition of $h_n(\overline{x})$.

The function f_n is nonnegative. Since $h_o(\overline{x}) = 0$ for all $\overline{x}$, it is evident that $f_1(\overline{x}) \geq 0$ for all $\overline{x}$. Consider $n > 1$. If $\rho(\overline{x}, \overline{0}) < n - 1$ and $f(\overline{x}) \leq n - 1$, then $h_n(\overline{x}) = h_{n-1}(\overline{x}) = f(\overline{x})$ so that $f_n(\overline{x}) = 0$. If $\rho(\overline{x}, \overline{0}) < n - 1$ and $n - 1 < f(\overline{x}) \leq n$, then

$$f_n(\overline{x}) = h_n(\overline{x}) - h_{n-1}(\overline{x}) = f(\overline{x}) - (n-1) > 0 .$$

If $\rho(\overline{x},\overline{0}) < n - 1$ and $f(\overline{x}) > n$, then $f_n(\overline{x}) = 0$. If $\rho(\overline{x},\overline{0}) \geq n - 1$, then $h_{n-1}(\overline{x}) = 0$ so that $f_n(\overline{x}) \geq 0$. Hence $f_n(\overline{x}) \geq 0$ for all $\overline{x}$.

Therefore, for each n it is true that f_n is Lebesgue measurable, nonnegative, bounded, and has a bounded support. Hence the argument in case 1 applies to each f_n, and a function g_n corresponding to f_n is obtained which satisfies conditions (i), (ii), and (iii) of the theorem. It will now be shown that the function defined by the series $\sum_{n=1}^{\infty} g_n(\overline{x})$ satisfies the conditions of the theorem in the general case. Let

$$g(\overline{x}) = \sum_{n=1}^{\infty} g_n(\overline{x}) \quad \text{for each} \quad \overline{x} \in R_q .$$

(i) The function g is L.S.C. on R_q. Since each g_n is nonnegative and $g(\overline{x}) = \sup_n \sum_{k=1}^{n} g_k(\overline{x})$, it follows that $g(\overline{x})$ is the supremum of a nonempty collection of L.S.C. functions. Hence, according to Lemma 2.3, $g(\overline{x})$ is also L.S.C. on R_q.

(ii) Also $g(\overline{x}) \geq f(\overline{x})$ for each $\overline{x} \in R_q$. This is verified by observing that

$$g(\overline{x}) = \sum_{n=1}^{\infty} g_n(\overline{x}) \geq \sum_{n=1}^{\infty} f_n(\overline{x}) = f(\overline{x})$$

for each $\overline{x} \in R_q$.

(iii) Finally, it must be shown that g satisfies condition (iii) of the theorem. Let $B = \left\{ \overline{x} \mid f(\overline{x}) < +\infty \right\}$ and note that B is Lebesgue measurable. Let $\epsilon > 0$ be given. By case 1 there exists a function g_n

defined on R_q such that

$$\int_{R_q} (g_n - f_n)\, d\mu < \frac{\epsilon}{2^n} \quad (n=1,2,\ldots).$$

Hence

$$\int_B (g-f)\, d\mu = \int_B \left[\sum_{n=1}^{\infty} (g_n - f_n)\right] d\mu = \sum_{n=1}^{\infty} \int_B (g_n-f_n)\, d\mu$$

$$\leq \sum_{n=1}^{\infty} \int_{R_q} (g_n-f_n)\, d\mu \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon .$$

Thus, if h is defined as in the note following the statement of the theorem, it follows that

$$\int_{R_q} h\, d\mu = \int_B (g-f)\, d\mu < \epsilon .$$

This completes the proof of the theorem. ∎

<u>Theorem 2.6</u>: (Theorem of Vitali-Carathéodory): Let $f : R_q \rightarrow R^*$ be a Lebesgue measurable function. Then there exist monotonic sequences $\{\ell_n\}$ and $\{u_n\}$ of functions, each function defined on $R_q \rightarrow R^*$ with the following properties:

(i) For each $n = 1,2,\ldots$ u_n is L.S.C. and ℓ_n is U.S.C.

(ii) Each function u_n is bounded below and each function ℓ_n is bounded above.

(iii) $\{\ell_n\}$ is nondecreasing and u_n is nonincreasing; i.e., $\ell_n(\overline{x})\uparrow$ and $u_n(\overline{x})\downarrow$ for each fixed $\overline{x} \in R_q$.

(iv) $\ell_n(\overline{x}) \leq f(\overline{x}) \leq u_n(\overline{x})$ for each $\overline{x} \in R_q$ and for $n=1,2,\ldots$.

(v) Except on a set of Lebesgue measure zero,

$$\lim_{n\to\infty} \ell_n(\overline{x}) = f(\overline{x}) = \lim_{n\to\infty} u_n(\overline{x}) .$$

(vi) On every Lebesgue measurable set A over which f is Lebesgue summable, so also is each ℓ_n and each u_n, and

$$\lim_{n\to\infty} \int_A \ell_n \, d\mu = \lim_{n\to\infty} \int_A u_n \, d\mu = \int_A f \, d\mu .$$

Proof: Case 1. Assume that f is nonnegative. Since f is nonnegative and Lebesgue measurable, an application of Lemma 2.5 yields a sequence of L.S.C. functions g_n such that $g_n(\overline{x}) \geq f(\overline{x})$ for each $\overline{x} \in R_q$, and

$$\int_{R_q} (g_n - f) \, d\mu < \frac{1}{2^n} \quad \text{for} \quad n=1,2,\ldots.$$

(it is understood that occurrences of $g_n(\overline{x}) - f(\overline{x}) = +\infty - \infty$ are replaced by 0). Moreover

$$\lim_{n\to\infty} \int_{R_q} (g_n - f) \, d\mu = 0 .$$

Define the functions u_n by

$$u_n(\overline{x}) = \inf \left\{ g_j(\overline{x}) \mid j = 1,2,\ldots,n \right\}$$

for each $\overline{x} \in R_q$. It will now be shown that the sequence $\left\{u_n\right\}$ serves as a sequence of u-functions of the theorem for the function f.

(i) Since u_n is the infimum of a finite collection of L.S.C.

functions defined on R_q, it is also L.S.C. on R_q.

(ii) Furthermore $u_n(\overline{x})$ is never less than zero on R_q. Otherwise some $g_j(\overline{x})$ would be less than zero for some $\overline{x}$ and some j. But $g_j(\overline{x})$ is nonnegative for each $\overline{x} \in R_q$. Thus $u_n(\overline{x})$ is bounded below by zero for every $n(n = 1,2,\ldots)$. That is, $u_n(\overline{x}) \geq 0$ for each $\overline{x} \in R_q$.

(iii) $u_n(\overline{x})$ is nonincreasing for each fixed $\overline{x} \in R_q$. This follows from the definition of u_n. In fact

$$u_{n+1}(\overline{x}) = \inf \left\{ g_j(\overline{x}) \mid j = 1,2,\ldots,n+1 \right\}$$

$$\leq \inf \left\{ g_j(\overline{x}) \mid j = 1,2,\ldots,n \right\} = u_n(\overline{x})$$

for each $\overline{x} \in R_q$ and for each $n = 1,2,\ldots$.

(iv) Since $g_n(\overline{x}) \geq f(\overline{x})$ for each $\overline{x} \in R_q$, it follows that $u_n(\overline{x}) \geq f(\overline{x})$ for each $\overline{x} \in R_q$ and for each $n = 1,2,\ldots$.

(v) Let $B = \left\{ \overline{x} \mid f(\overline{x}) < +\infty \right\}$. Then it follows that

$$0 \leq \int_B (g_n - f)\, d\mu < \frac{1}{2^n}$$

so that

$$\lim_{n\to\infty} \int_B (g_n - f)\, d\mu = 0$$

Hence

$$\lim_{n\to\infty} \int_B (u_n - f)\, d\mu = 0.$$

Now, by the Lebesgue Monotone Convergence Theorem, it follows that $\lim_{n\to\infty} u_n(\overline{x}) = f(\overline{x})$ a.e. on B (relative to Lebesgue measure μ). If

$\overline{x} \in R_q - B$, $u_n(\overline{x}) = f(\overline{x}) = +\infty$, so that $\lim\limits_{n\to\infty} u_n(\overline{x}) = f(\overline{x})$ everywhere on $R_q - B$. Thus

$$\lim_{n\to\infty} u_n(\overline{x}) = f(\overline{x}) \text{ a.e. on } R \text{ (relative to } \mu).$$

(vi) Let A be a Lebesgue measurable set in R_q and let f be Lebesgue summable over A. In the calculation below the function f defined on A will be replaced by the function $\tilde{f}$ which is everywhere finite on A with $\int_A \tilde{f}\, d\mu = \int_A f\, d\mu$. Since g_n was chosen so that

$$\int_{R_q} (g_n - f)\, d\mu < \frac{1}{2^n} \text{ for } n = 1,2,\ldots,$$

it follows that

$$\int_{R_q} (u_n - f)\, d\mu < \frac{1}{2^n} .$$

Thus

$$\int_A (u_n-\tilde{f})\, d\mu = \int_A (u_n-f)\, d\mu < \frac{1}{2^n} \ (n = 1,2,\ldots) ,$$

and

$$\int_A u_n\, d\mu < \frac{1}{2^n} + \int_A \tilde{f}\, d\mu < +\infty .$$

Hence each u_n is Lebesgue summable over A $(n = 1,2,\ldots)$. Thus

$$\lim_{n\to\infty} \int_A (u_n-\tilde{f})\, d\mu = 0$$

which yields

$$\lim_{n\to\infty} \int_A u_n \, d\mu = \int_A \tilde{f} \, d\mu = \int_A f \, d\mu .$$

To define a sequence $\langle \ell_n \rangle$ of ℓ-functions for f, consider the reciprocal function $\frac{1}{f}$ with the understanding that $\frac{1}{f(\bar{x})} = 0$ if $f(\bar{x}) = +\infty$ and $\frac{1}{f(\bar{x})} = +\infty$ if $f(x) = 0$. By the first part of the argument, the existence of a nonincreasing sequence of L.S.C. functions $\langle h_n \rangle$ is deduced such that $\langle h_n \rangle$ satisfies the six requirements of the theorem as u-functions relative to the function $\frac{1}{f}$. Since h_n is L.S.C. on R_q, Lemma 2.4 implies that $\frac{1}{h_n}$ is U.S.C. on R_q. Hence $\left\langle \frac{1}{h_n} \right\rangle$ is a nondecreasing sequence of U.S.C. functions defined on R_q. Also $\lim_{n\to\infty} h_n(\bar{x}) = \frac{1}{f(\bar{x})}$ a.e. on R_q (relative to Lebesgue measure μ). Thus $\lim_{n\to\infty} \frac{1}{h_n(\bar{x})} = f(\bar{x})$ a.e. in R_q (relative to μ). For each $n = 1,2,\ldots$ define the function ℓ_n by:

$$\begin{aligned} \ell_n(\bar{x}) &= \frac{1}{h_n(\bar{x})}, \quad \text{if } \frac{1}{h_n(\bar{x})} \leq n \quad \left(\tfrac{1}{\infty} \text{ is replaced by } 0 \right); \\ &= n \quad\quad\;\, , \quad \text{if } \frac{1}{h_n(\bar{x})} > n \quad \left(\tfrac{1}{0} \text{ is replaced by } +\infty \right). \end{aligned}$$

Thus a sequence $\langle \ell_n \rangle$ of functions has been obtained which serves as the sequence of ℓ-functions of the theorem for the function f. It will now be shown that the sequence $\langle \ell_n \rangle$ satisfies the six requirements of the theorem for ℓ-functions.

(i) h_n is L.S.C. on R_q $(n = 1,2,\ldots)$ and hence $\frac{1}{h_n}$ is U.S.C. on R_q. At each $\bar{x} \in R_q$ the function ℓ_n is U.S.C. since $\ell_n(\bar{x})$ is either equal to $\frac{1}{h_n(\bar{x})}$, which is U.S.C., or it is equal to n, which is likewise U.S.C. on R_q.

(ii) Each ℓ_n is bounded above; i.e., $\ell_n(\bar{x}) \leq n$ for each $\bar{x} \in R_q$ $(n = 1,2,\ldots)$.

(iii) Since $\{h_n(\bar{x})\}$ is nonincreasing, $\left\{\frac{1}{h_n(\bar{x})}\right\}$ is nondecreasing for each $\bar{x} \in R_q$. Thus $\{\ell_n\}$ is nondecreasing on R_q.

(iv) For each $n = 1,2,\ldots$ it is true that $h_n(\bar{x}) \geq \frac{1}{f(\bar{x})}$ for each $\bar{x} \in R_q$. Moreover, $\frac{1}{h_n(x)} \leq f(\bar{x})$ and thus $\ell_n(\bar{x}) \leq \frac{1}{h_n(\bar{x})} \leq f(\bar{x})$ for each $\bar{x} \in R_q$ $(n = 1,2,\ldots)$.

(v) Let the sets S_1 and S_2 be defined by:

$$S_1 = \left\{ \bar{x} \mid \bar{x} \in R_q,\ \lim_{n\to\infty} \frac{1}{h_n(\bar{x})} \text{ is finite} \right\} ;$$

and

$$S_2 = \left\{ \bar{x} \mid \bar{x} \in R_q,\ \lim_{n\to\infty} \frac{1}{h_n(\bar{x})} = +\infty \right\} .$$

Note that $S_1 \cup S_2 = R_q$. If $\bar{x} \in S_1$,

$$\lim_{n\to\infty} \ell_n(\bar{x}) = \lim_{n\to\infty} \frac{1}{h_n(\bar{x})} = f(\bar{x}) \quad \text{a.e. in } S_1 \text{ (relative to } \mu) .$$

Furthermore, for almost every $\bar{x} \in S_2$,

$$\lim_{n\to\infty} \ell_n(\bar{x}) = \lim_{n\to\infty} n = \lim_{n\to\infty} \frac{1}{h_n(\bar{x})} = f(\bar{x}) .$$

Hence

$$\lim_{n\to\infty} \ell_n(\bar{x}) = f(\bar{x}) \quad \text{a.e. in } R_q \quad \text{(relative to } \mu) .$$

(vi) Let A be a Lebesgue measurable set over which f is Lebesgue summable. Now $\{\ell_n\}$ is a nonnegative nondecreasing sequence of Lebesgue measurable functions defined on $A \subset R_q$ such that $\lim_{n\to\infty} \ell_n(\overline{x}) = f(\overline{x})$ a.e. in A (relative to Lebesgue measure μ). Furthermore, each ℓ_n satisfies $\ell_n(\overline{x}) \leq f(\overline{x})$ for each $\overline{x} \in R_q$ and for each $n = 1,2,\ldots$. The Lebesgue Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_A \ell_n \, d\mu = \int_A f \, d\mu .$$

Moreover, since f is Lebesgue summable over A,

$$\int_A \ell_n \, d\mu \leq \int_A f \, d\mu < +\infty \text{ for each } n = 1,2,\ldots .$$

This proves that each ℓ_n is Lebesgue summable over A and completes the discussion of case 1.

Case 2. Assume that the hypotheses of the theorem hold. Let $f : R_q \rightarrow R^*$ be Lebesgue measurable. As is used in the theory of the integral write $f = f^+ - f^-$. Consider the nonnegative measurable functions f^+ and f^-. In particular, associated with f^+ are the two sequences $\{\ell_n^+\}$ and $\{u_n^+\}$ and with the function f^- the two sequences $\{\ell_n^-\}$ and $\{u_n^-\}$. Thus

$$\ell_n^+(\overline{x}) \leq f^+(\overline{x}) \leq u_n^+(\overline{x})$$

and

$$\ell_n^-(\overline{x}) \leq f^-(\overline{x}) \leq u_n^-(\overline{x})$$

for each $\overline{x} \in R_q$ and each $n = 1,2,\ldots$. Hence

$$\ell_n^+(\overline{x}) \leq f^+(\overline{x}) \leq u_n^+(\overline{x}) \quad ,$$

$$-u_n^-(\overline{x}) \leq -f^-(\overline{x}) \leq -\ell_n^-(\overline{x}) \ ,$$

and

$$\ell_n^+(\overline{x}) - u_n^-(\overline{x}) \leq f(\overline{x}) \leq u_n^+(\overline{x}) - \ell_n^-(\overline{x}) \ .$$

Define the functions ℓ_n and u_n by

$$\ell_n = \ell_n^+ - u_n^- \text{ and } u_n = u_n^+ - \ell_n^-$$

for each $n = 1,2,\ldots$. It will now be shown that $\langle \ell_n \rangle$ and $\langle u_n \rangle$ satisfy the six requirements of the theorem.

(i) For each $n = 1,2,\ldots$ ℓ_n^- is U.S.C. on R_q. Thus $-\ell_n^-$ is L.S.C. on R_q (Theorem 1, Appendix). Furthermore, u_n^+ is L.S.C. on R_q so that the sum $u_n^+ - \ell_n^-$ is L.S.C. on R_q. Thus each u_n is L.S.C. on R_q. Similarly it is shown that each ℓ_n is U.S.C. on R_q $(n = 1,2,\ldots)$.

(ii) Since $u_n^+(\overline{x}) \geq 0$ and $-\ell_n^-(\overline{x}) \geq -n$ for each $x \in R_q (n = 1,2,\ldots)$, it follows that

$$u_n(\overline{x}) = u_n^+(\overline{x}) - \ell_n^-(\overline{x}) \geq -n \ .$$

Thus each function u_n is bounded below. Likewise $\ell_n^+(\overline{x}) \leq n$ and $-u_n^-(\overline{x}) \leq 0$ for each $\overline{x} \in R_q (n = 1,2,\ldots)$, so that

$$\ell_n(\overline{x}) = \ell_n^+(\overline{x}) - u_n^-(\overline{x}) \leq n \ .$$

This implies that each function ℓ_n is bounded above.

(iii) Since $u_n^-(\overline{x})\downarrow$, then $-u_n^-(\overline{x})\uparrow$. Consequently $\ell_n(\overline{x})\uparrow$ for each fixed $\overline{x} \in R_q$. Also $\ell_n^-(\overline{x})\uparrow$ so $-\ell_n^-(x)\downarrow$, and it follows that $u_n(\overline{x}) \downarrow$ for each fixed $\overline{x} \in R_q$.

(iv) By the definition of ℓ_n and u_n, it follows that

$$\ell_n(\overline{x}) \leq f(\overline{x}) \leq u_n(\overline{x})$$

for every $\overline{x} \in R_q$ $(n = 1,2,\ldots)$.

(v) Define the sets B_1 and B_2 in R_q by:

$$B_1 = \left\{ \overline{x} \mid \lim_{n\to\infty} \ell_n^+ (\overline{x}) \neq f^+ (\overline{x}) \right\};$$

$$B_2 = \left\{ \overline{x} \mid \lim_{n\to\infty} u_n^- (\overline{x}) \neq f^- (\overline{x}) \right\}.$$

Since $\lim_{n\to\infty} \ell_n^+ (\overline{x}) = f^+ (\overline{x})$ a.e. in R_q (relative to Lebesgue measure μ) and $\lim_{n\to\infty} u_n^- (\overline{x}) = f^- (x)$ a.e. in R_q (relative to Lebesgue measure μ), it follows that $\mu(B_1) = \mu(B_2) = 0$. Choose $\overline{x} \in R_q - (B_1 \bigcup B_2)$ such that $f(\overline{x})$ is not finite. Then

$$\lim_{n\to\infty} \ell_n(\overline{x}) = \lim_{n\to\infty} \left[\ell_n^+ (\overline{x}) - u_n^- (\overline{x}) \right]$$

$$= \lim_{n\to\infty} \ell_n^+ (\overline{x}) - \lim_{n\to\infty} u_n^- (\overline{x}) = f(\overline{x}) .$$

Now suppose that $\overline{x} \in R_q - (B_1 \bigcup B_2)$ with $f(\overline{x})$ finite. Then

$$\lim_{n\to\infty} \ell_n(\overline{x}) = \lim_{n\to\infty} \left[\ell_n^+ (\overline{x}) - u_n^- (\overline{x}) \right] = \lim_{n\to\infty} \ell_n^+ (\overline{x}) - \lim_{n\to\infty} u_n^- (\overline{x}) = f(\overline{x}).$$

Hence

$$\lim_{n\to\infty} \ell_n(\overline{x}) = f(\overline{x})$$

for each $\overline{x} \in R_q - (B_1 \bigcup B_2)$ or a.e. in R_q (relative to Lebesgue measure μ). By a similar argument

$$\lim_{n\to\infty} u_n(\overline{x}) = f(\overline{x})$$

a.e. on R_q (relative to μ).

(vi) Let A be a Lebesgue measurable set in R_q over which f is Lebesgue summable. By the Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty} \int_A \ell_n \, d\mu = \int_A f \, d\mu \; .$$

Also, as a consequence of the Monotone Convergence Theorem, it is true that

$$\lim_{n\to\infty} \int_A u_n \, d\mu = \int_A f \, d\mu \; .$$

For each $n = 1,2,\ldots$ the functions ℓ_n^+, ℓ_n^-, u_n^+, and u_n^- are Lebesgue summable over A by case 1. This implies the Lebesgue summability of u_n and ℓ_n over A for each $n = 1,2,\ldots$. This completes the proof of the theorem. ∎

Lemma 2.7: Let D be a nonempty subset of R_q and let $f : D \to R^*$ be lower semicontinuous on D. If there exists a finite number M such that $f(\overline{x}) \geq M$ for each $\overline{x} \in D$, then there exists a sequence of continuous functions $\Phi_n : R_q \to R$ $(n = 1,2,\ldots)$ such that
$M \leq \Phi_1(\overline{x}) \leq \Phi_2(\overline{x}) \ldots$, and

$$\lim_{n\to\infty} \Phi_n(\overline{x}) = f(\overline{x}) \quad \text{for all } \overline{x} \in D \; .$$

Proof: Case 1. First assume that f is nonnegative on D and is finite for at least one point $\overline{x} \in D$ (if $f(\overline{x}) = +\infty$ for all $\overline{x} \in D$, the theorem is trivial). For each positive integer n and for each $\overline{x} \in R_q$, define the function Φ_n by

$$\Phi_n(\overline{x}) = \inf\left\{ f(\overline{y}) + n\,\rho(\overline{y},\overline{x}) \;\middle|\; \overline{y} \in D \right\}$$

where $\rho(\overline{y},\overline{x}) = \|\overline{x}-\overline{y}\|$. $\Phi_n(\overline{x})$ is obviously finite for each $\overline{x} \in R_q$ ($n = 1,2,\ldots$).

Suppose $\overline{x}'$ and $\overline{x}''$ are in R_q. Then, for all $\overline{y} \in D$,

$$\Phi_n(\overline{x}') \leq f(\overline{y}) + n\left[\rho(\overline{y},\overline{x}'') + \rho(\overline{x}'',\overline{x}')\right] ,$$

and it follows that

$$\Phi_n(\overline{x}') - n\,\rho(\overline{x}'',\overline{x}') \leq f(\overline{y}) + n\,\rho(\overline{y},\overline{x}'')$$

for all $\overline{y} \in D$. Hence

$$\Phi_n(\overline{x}') - n\,\rho(\overline{x}'',\overline{x}') \leq \Phi_n(\overline{x}'')$$

since $\Phi_n(\overline{x}'')$ is the infimum of numbers of this form. Furthermore, by interchanging $\overline{x}'$ and $\overline{x}''$ the analogous inequality

$$\Phi_n(\overline{x}'') - n\,\rho(\overline{x}'',\overline{x}') \leq \Phi_n(\overline{x}')$$

is obtained. Consequently

$$\left|\Phi_n(\overline{x}') - \Phi_n(\overline{x}'')\right| \leq n\,\rho(\overline{x}';\overline{x}'') \quad ,$$

and the function Φ_n is continuous on R_q for each $n = 1,2,\ldots$.

By the definition of Φ_n, it follows that $\Phi_n(\overline{x}) \leq \Phi_{n+1}(\overline{x})$ for all $\overline{x} \in R_q$ ($n = 1,2,\ldots$). Also for each $\overline{x} \in D$ and for each $n = 1,2,\ldots$,

$$\Phi_n(\overline{x}) = \inf\left\{f(\overline{y}) + n\ \rho(\overline{x},\overline{y}) \mid \overline{y} \in D\right\}$$

$$\leq f(\overline{x}) + n\ \rho(\overline{x},\overline{x}) = f(\overline{x}) \ .$$

Hence

$$\lim_{n\to\infty} \Phi_n(\overline{x}) \leq f(\overline{x}) \quad \text{for each} \quad \overline{x} \in D \ .$$

On the other hand, if h is any number less than $f(\overline{x})$, then there corresponds a $\delta > 0$ such that $f(\overline{y}) > h$ provided $y \in N(\overline{x};\delta) \cap D$ (Theorem 2, Appendix). Let n be any integer greater than $\frac{h}{\delta}$. If $\overline{y} \in D$, either $\rho(\overline{y},\overline{x}) < \delta$ and

$$f(\overline{y}) + n\ \rho(\overline{y},\overline{x}) \geq f(\overline{y}) > h \ ,$$

or $\rho(\overline{y},\overline{x}) \geq \delta$ and

$$f(\overline{y}) + n\ \rho(\overline{y},\overline{x}) > f(\overline{y}) + h \geq h \ .$$

Hence h is a lower bound for $f(\overline{y}) + n\ \rho(\overline{y},\overline{x})$ for all $\overline{y} \in D$. Thus $\Phi_n(\overline{x}) \geq h$ and this implies that $\lim_{n\to\infty} \Phi_n(\overline{x}) \geq h$. But, since h is any number less than $f(\overline{x})$, it must be true that $\lim_{n\to\infty} \Phi_n(\overline{x}) = f(\overline{x})$ for each $\overline{x} \in D$. This completes the proof of case 1.

Case 2. Now suppose that the hypotheses of the lemma are satisfied in the form stated. Since $f(\overline{x}) \geq M$, then $f(\overline{x}) - M \geq 0$. Consider the function $f - M$. It is L.S.C. and nonnegative on D. Thus, according to case 1, there exist continuous functions $\Phi_n : R_q \longrightarrow R$ $(n = 1,2,\ldots)$ such that $\overline{\Phi}_1(\overline{x}) \leq \overline{\Phi}_2(\overline{x}) \leq \ldots$, and $\lim_{n\to\infty} \overline{\Phi}_n(\overline{x}) = f(\overline{x}) - M$ for all $\overline{x} \in D$.

Now, define for each $n = 1,2,\ldots,$ the function Φ_n by

$$\Phi_n(\overline{x}) = \overline{\Phi}_n(\overline{x}) + M \quad \text{for each} \quad \overline{x} \in R_q .$$

Clearly

$$M \leq \Phi_1(\overline{x}) \leq \Phi_2(\overline{x}) \leq \cdots$$

for every $\overline{x} \in D$ since $\overline{\Phi}_n(\overline{x}) \geq 0$ for each $n = 1,2,\ldots$ and for all $\overline{x} \in D$. Moreover,

$$\lim_{n\to\infty} \Phi_n(\overline{x}) = \lim_{n\to\infty} (\overline{\Phi}_n + M) = f(\overline{x})$$

for each $\overline{x} \in D$ and for each $n = 1,2,\ldots$. This completes the proof of the lemma. ∎

CHAPTER III

CHANGE OF VARIABLE THEOREMS FOR THE LEBESGUE INTEGRAL IN THE ONE-DIMENSIONAL CASE

In this chapter two transformation formulas will be proved which describe the effect of a change of variable in a Lebesgue integral in the real line. The theorems and the lemmas of the preceding chapter are essential to the proofs. The first theorem is concerned with a change of variable in the Lebesgue integral of a bounded Lebesgue measurable function f defined on the closed interval $[a,b]$ in R. The second theorem is similar, except that the function f is assumed to be Lebesgue summable over the closed interval $[a,b]$.

Theorem 3.1: Let $f : [a,b] \rightarrow R$ be bounded and Lebesgue summable, and let $u : [c,d] \rightarrow [a,b]$ be absolutely continuous on $[c,d]$. Further let $\mathring{u}$ be a function defined on $[c,d]$ equal to the derivative of u whenever the derivative exists and is finite. Then the function $f[u(t)]\mathring{u}(t)$, defined on $[c,d]$, is Lebesgue summable over $[c,d]$, and

$$(1) \qquad \int_{u(c)}^{u(d)} f(x)\, dx = \int_c^d f[u(t)]\, \mathring{u}(t)\, dt .$$

Note: $\mathring{u}(t)$ is defined by:

$$\begin{aligned} \mathring{u}(t) &= u'(t), \quad \text{for } t \in [c,d] \text{ whenever } u'(t) \\ &\qquad \text{is defined and finite;} \\ &= 0 \quad , \text{ otherwise.} \end{aligned}$$

Proof: It should be remarked that f is necessarily Lebesgue summable on $[a,b]$ since f is bounded and Lebesgue measurable on $[a,b]$. Now define the function F by

$$F(t) = \int_a^t f(x)\,dx \quad \text{for all} \quad t \in [a,b] .$$

The left side of equation (1) is therefore $F[u(d)] - F[u(c)]$. Thus, it must be shown that

$$F[u(d)] - F[u(c)] = \int_c^d f[u(t)]\dot{u}(t)\,dt .$$

Define the function G by $G(t) = F[u(t)]$ for each $t \in [c,d]$. If K is an upper bound for $|f(x)|$ on $[a,b]$, then it follows that F satisfies a uniform Lipschitz condition on $[a,b]$ since

$$-K(x_2-x_1) \leq \int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1) \leq K(x_2-x_1)$$

whenever $a \leq x_1 \leq x_2 \leq b$. Furthermore, since f is Lebesgue summable over $[a,b]$, the function F is absolutely continuous on $[a,b]$. Hence, the function $G(t) = F[u(t)]$ is absolutely continuous (Theorem 3, Appendix).

Case 1. At the first stage in the proof, suppose that f is continuous on $[a,b]$. Then the derivative F' exists and is equal to f everywhere in $[a,b]$. It follows from the absolute continuity of u that u' exists and is finite almost everywhere in $[c,d]$. Thus $\mathring{u}(t) = u'(t)$ a.e. in $[c,d]$ (relative to Lebesgue measure μ) and is equal to zero on the set of Lebesgue measure zero where $u'(t)$ does not exist or exists and is infinite. Since $F'(x) = f(x)$ for all $x \in [a,b]$,

$F'[u(t)] = f[u(t)]$ for all t in $[c,d]$, and thus, using the chain rule, $G'(t) = F'[u(t)]\mathring{u}(t) = f[u(t)]\mathring{u}(t)$ a.e. in the interval $[c,d]$. G is absolutely continuous on $[c,d]$, and thus G' is Lebesgue summable over $[c,d]$. It follows that

$$G(d) - G(c) = \int_c^d \mathring{G}(t)\, dt ,$$

where $\mathring{G}(t)$ is defined analogously to $\mathring{u}$. Hence

$$G(d) - G(c) = \int_c^d \mathring{G}(t)\, dt = \int_c^d G'(t)\, dt = \int_c^d f[u(t)]\mathring{u}(t)\, dt .$$

But, by definition, $G(d) - G(c) = F[u(d)] - F[u(c)]$, and thus Equation (1) is verified for a continuous function f. Hence

$$\int_{u(c)}^{u(d)} f(x)\, dx = \int_c^d f[u(t)]\, \mathring{u}(t)\, dt .$$

Case 2. It remains to discuss the general case. Let the hypotheses of the theorem hold. Consider the bounded and measurable function $f : [a,b] \rightarrow R$ (recall that f is Lebesgue summable over $[a,b]$). By the Theorem of Vitali-Carathéodory (Theorem 2.6), there exists a sequence $\{u_j\}$ defined on $[a,b]$ with the following properties:

(i) For each $j = 1,2,\ldots$ u_j is L.S.C. on $[a,b]$.

(ii) Each function u_j is bounded below on $[a,b]$ $(j = 1,2,\ldots)$.

(iii) u_j is nonincreasing on $[a,b]$. That is,

$$u_1(x) \geq u_2(x) \geq \cdots \geq u_j(x) \geq \cdots \quad \text{for each } x \in [a,b].$$

(iv) $u_j(x) \geq f(x)$ for each $j = 1,2,\ldots$ and for every $x \in [a,b]$.

(v) $\lim_{j\to\infty} u_j(x) = f(x)$ a.e. in $[a,b]$ (relative to μ).

(vi) Since f is Lebesgue summable over $[a,b]$, so is each $u_j (j = 1,2,\ldots)$, and

$$\lim_{j\to\infty} \int_a^b u_j(x)\,dx = \int_a^b f(x)\,dx .$$

Now, for each $j = 1,2,\ldots,$ define the function g_j by

$$g_j(x) = \inf\left\{u_j(x), K\right\} \text{ for every } x \text{ in } [a,b],$$

where K is an upper bound for $|f(x)|$. It is easily verified that the sequence $\left\{g_j(x)\right\}$ satisfies the six conditions of the Vitali-Caratheodory Theorem. In particular, $g_j\colon [a,b] \to R$ in L.S.C. on $[a,b]$, and there exists a number M_j such that $M_j \leq g_j(x)$ for each x in $[a,b]$ and for each $j = 1,2,\ldots$. Consequently, for each $j = 1,2,\ldots,$ there exists a sequence of continuous functions $\Phi_{jn}\colon [a,b] \to R$ $(n = 1,2,\ldots)$ such that

$$M_j \leq \Phi_{j1}(x) \leq \Phi_{j2}(x) \leq \cdots \leq g_j(x)$$

and

$$\lim_{n\to\infty} \Phi_{jn}(x) = g_j(x)$$

for each x in $[a,b]$ and for each fixed $j = 1,2,\ldots$. From the argument in case 1, it follows that Equation (1) holds for each function Φ_{jn}. Thus

$$\int_{u(c)}^{u(d)} \Phi_{jn}(x)\,dx = \int_c^d \Phi_{jn}\,[u(t)]\,\dot{u}(t)\,dt . \tag{2}$$

Since for each $j = 1,2,\ldots,$

$$M_j \leq \Phi_{j1}(x) \leq \Phi_{j2}(x) \leq \cdots \leq g_j(x) \leq K ,$$

it follows that $|\Phi_{jn}(x)| \leq K + |M_j|$ for each x in $[a,b]$ and for $n = 1,2,\ldots$. And, since $\mathring{u}$ is Lebesgue summable on $[c,d]$, the functions $\Phi_{jn}[u(t)]\ \mathring{u}(t)$ do not exceed the Lebesgue summable function $(K + |M_j|)|\mathring{u}|$ is absolute value $(n = 1,2,\ldots)$. Hence, if the limit of both sides of Equation (2) is taken and if the Lebesgue Dominated Convergence Theorem is applied, it follows that

$$\int_{u(c)}^{u(d)} g_j(x)\ dx = \int_c^d g_j[u(t)]\ \mathring{u}(t)\ dt \quad (j = 1,2,\ldots).$$

Again, by an application of the Lebesgue Dominated Convergence Theorem and condition (vi) regarding the sequence $\{g_j(x)\}$, it follows that

$$\lim_{j\to\infty} \int_{u(c)}^{u(d)} g_j(x)\ dx = \lim_{j\to\infty} \int_c^d g_j[u(t)]\ \mathring{u}(t)\ dt$$

and

$$\int_{u(c)}^{u(d)} f(x)\ dx = \int_c^d f[u(t)]\ \mathring{u}(t)\ dt .$$

This completes the proof of the theorem. ∎

<u>Theorem 3.2</u>: Let $f : [a,b] \rightarrow R^*$ be Lebesgue summable, and let $u : [c,d] \rightarrow [a,b]$ be monotonic and absolutely continuous on $[c,d]$. Furthermore, let $\mathring{u}$ be a function, defined on $[c,d]$, equal to the derivative of u whenever the derivative exists and is finite. Then the function defined by $f[u(t)]\ \mathring{u}(t)$ is Lebesgue summable over $[c,d]$, and

$$(3) \qquad \int_{u(c)}^{u(d)} f(x)\,dx = \int_{c}^{d} f[u(t)]\,\mathring{u}(t)\,dt\ .$$

Note: The function $\mathring{u}$ is defined exactly as in Theorem 3.1.

Proof: It suffices to consider the case where u is monotonically increasing; if u is monotonically decreasing, the following proof will apply with only minor changes.

Suppose first that f is nonnegative on $[a,b]$. For each $n = 1,2,\ldots$ define the function f_n by:

$$\begin{aligned} f_n(x) &= f(x), \quad \text{if} \quad f(x) < n\ ; \\ &= n\ , \quad \text{if} \quad f(x) \geq n\ . \end{aligned}$$

Then, for each $n = 1,2,\ldots,$ it follows from Theorem 3.1 that

$$(4) \qquad \int_{u(c)}^{u(d)} f_n(x)\,dx = \int_{c}^{d} f_n[u(t)]\,\mathring{u}(t)\,dt\ .$$

Note that $\{f_n\}$ is an increasing sequence of nonnegative Lebesgue measurable functions such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for each $x \in [a,b]$. Moreover, u is absolutely continuous and monotonically increasing, so that $\mathring{u}(t) \geq 0$ for each $t \in [c,d]$. Thus the sequence $\{f_n[u(t)]\,\mathring{u}(t)\}$ is also an increasing sequence of nonnegative Lebesgue measurable functions. If the limit of both sides of equation (4) is taken as $n\to\infty$ and the Monotone Convergence Theorem is applied, then

$$\lim_{n\to\infty} \int_{u(c)}^{u(d)} f_n(x)\,dx = \lim_{n\to\infty} \int_{c}^{d} f_n[u(t)]\,\mathring{u}(t)\,dt\ ,$$

and

$$\int_{u(c)}^{u(d)} f(x)\,dx = \int_c^d f[u(t)]\,\mathring{u}(t)\,dt .$$

Thus $f[u(t)]\,\mathring{u}(t)$ is Lebesgue summable over $[c,d]$ since f is Lebesgue summable over $[a,b]$.

It is necessary now to drop the assumption that f is nonnegative on $[a,b]$. In the general case f can be written as the difference of two nonnegative Lebesgue summable functions, $f = f^+ - f^-$. If the preceding results are applied to f^+ and f^-, the equations

$$\int_{u(c)}^{u(d)} f^+(x)\,dx = \int_c^d f^+\,[u(t)]\,\mathring{u}(t)\,dt \tag{5}$$

and

$$\int_{u(c)}^{u(d)} f^-(x)\,dx = \int_c^d f^-\,[u(t)]\,\mathring{u}(t)\,dt \tag{6}$$

are obtained. Equation (6) is subtracted from Equation (5) to obtain the desired result. Hence

$$\int_{u(c)}^{u(d)} f(x)\,dx = \int_c^d f[u(t)]\,\mathring{u}(t)\,dt . \tag{7}$$

Remark: If the agreement is made that $u'(t)$ be set equal to zero when $u'(t)$ does not exist or when $u'(t)$ exists and is infinite, then Equation (7) takes the form

$$\int_{u(c)}^{u(d)} f(x)\,dx = \int_c^d f[u(t)]u'(t)\,dt .$$

This completes the proof of the theorem. ∎

CHAPTER IV

TRANSFORMATION THEORY IN q-DIMENSIONAL EUCLIDEAN SPACE

In all that follows real-valued mappings of q real variables will be considered. The domain of the transformation as well as the range will always be sets of points in R_q (q-dimensional Euclidean space). As in the previous chapters a point or vector in R_q will be denoted by $\overline{x}$, where $\overline{x}$ is the ordered q-tuple of real numbers $\overline{x} = (x_1, x_2, \ldots, x_q)$. Many of the results which follow are also valid in abstract spaces.

A mapping f defined on a subset of R_q is defined to be an ordered q-tuple of real-valued functions:

$$f(\overline{x}) = (f_1(\overline{x}),\ f_2(\overline{x}),\ \ldots,\ f_q(\overline{x}))\ .$$

A mapping f, its domain $D \subset R_q$, and its range $D' \subset R'_q$ will be indicated by the symbol $f : D \rightarrow D'$. Thus, if $\overline{x} \in D$ and $\overline{y} = f(\overline{x}) \in D'$, then the mapping f may be characterized in the following way:

$$\overline{y} = f(\overline{x});\ \text{or}\ \ (y_1, y_2, \ldots, y_q) = (f_1(\overline{x}), f_2(\overline{x}), \ldots, f_q(x))\ ,$$

or even more explicitly by

$$y_j = f_j(x_1, x_2, \ldots, x_q)\ \text{ for each }\ j = 1, 2, \ldots, q\ .$$

Further, let M be an arbitrary subset of D, where D is the domain of f. Consider the set of all points $\overline{y}$ having the form $f(\overline{x})$, for $\overline{x} \in M$, and denote this set by $f(M)$. It follows that $f(M)$, for $M \subset D$, consists exactly of those points $\overline{y}$ for which the equation $\overline{y} = f(\overline{x})$ has a solution in the set M. The set $M' = f(M)$ is called the <u>image</u> of M, and M is called the <u>inverse image</u> of the set $M' = f(M)$.

<u>Continuous Mappings and Measurable Mappings</u>

<u>Definition 4.1</u>: <u>Continuity of a Mapping</u>. Let $f : D \rightarrow D'$ be a mapping and let $\overline{x}_o \in D$ with $f(\overline{x}_o) \in D'$. Then f is said to be continuous at $\overline{x}_o \in D$ if and only if for every sequence $\left\{\overline{x}_j\right\}$ in D converging to $\overline{x}_o$ the corresponding sequence $\left\{f(\overline{x}_j)\right\}$ in D' converges to $f(\overline{x}_o)$. In symbols $\lim_{j\to\infty} f(\overline{x}_j) = f(\overline{x}_o)$. If the mapping f is continuous at each point in D, then f is said to be <u>continuous on D</u>.

<u>Definition 4.2</u>: <u>Measurability of a Mapping</u>. Let $f : D \rightarrow D'$ be a continuous mapping of a Lebesgue measurable set $D \subset R_q$ into a set $D' \subset R_q'$. The mapping f is said to be <u>Lebesgue measurable</u> if and only if the image set M' of every Lebesgue measurable subset M of D is itself Lebesgue measurable.

<u>Definition 4.3</u>: <u>Property (N) (Null set preserving property)</u>. Given a mapping $f : D \rightarrow D'$. Let $N \subset D$ be a set of Lebesgue measure zero. If the image set $f(N)$ is also a set of Lebesgue measure zero, then the mapping f is said to possess property (N). Such sets N are commonly called "null sets" relative to Lebesgue measure, or "Lebesgue null sets."

It is now natural to ask for necessary and sufficient conditions for the Lebesgue measurability of an arbitrary continuous mapping f.

This question is answered in the following theorem.

<u>Theorem 4.4</u>: Let $f : D \to D'$ be a continuous mapping of a Lebesgue measurable set $D \subset R_q$ into a set $D' \subset R'_q$. A necessary and sufficient condition for the mapping f to be Lebesgue measurable is that f possess property (N).

<u>Proof</u>: Let M be an arbitrary Lebesgue measurable subset of D. Then there exists a set of type F_σ such that $F_\sigma \subset M$ and $\mu(M - F_\sigma) = 0$, where μ is Lebesgue measure in R_q. Hence, $M - F_\sigma = N$, a null set, and M can be written as the union of an F_σ and a null set. In particular, $M = F_\sigma \cup N$.

Further $F_\sigma = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed, but not necessarily bounded. However, each $F_n \subset R_q$ and R_q can be represented as a countable union of compact sets. Hence, F_n can be represented as a countable union of compact sets. It follows that $F_n = \bigcup_{j=1}^{\infty} F_{nj}$. Thus

$$F_\sigma = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} F_{nj} = \bigcup_{k=1}^{\infty} F_k ,$$

where each F_k is compact. But, since f is a continuous mapping,

$$f\left[\bigcup_{k=1}^{\infty} F_k\right] = \bigcup_{k=1}^{\infty} f(F_k) ,$$

where each $f(F_k)$ is compact. Therefore $f(F_\sigma)$ is also a set of type F_σ and is thus Lebesgue measurable.

Conversely, suppose that f is Lebesgue measurable on D. Let N be a null set in D. It will now be shown that $\mu[f(N)] = 0$. Assume that $\mu[f(N)] > 0$; according to a standard theorem from measure theory (cf. [8]),

$f(N)$ contains a Lebesgue non-measurable subset $f(N_1)$ which is the image of a subset N_1 of N. But N_1 is a subset of a null set and is necessarily Lebesgue measurable since Lebesgue measure is complete. Furthermore, since f was assumed to be Lebesgue measurable, $f(N_1)$ <u>must</u> be Lebesgue measurable; hence, a contradiction is obtained which implies $\mu[f(N)] = 0$. Thus $f(N)$ is a null set and the proof is complete. ∎

A mapping $f : D \rightarrow D'$ of a Lebesgue measurable set D into a set D' also induces a mapping of the family of all subsets of the set D into the family of all subsets of D'. Thus f can be regarded as a generalized set function. The following definition of absolute continuity of a generalized set function will be essential for later work.

<u>Definition 4.5</u>: Let $f : D \rightarrow D'$ be a generalized set function defined on the family of all subsets of D, where D is Lebesgue measurable. Let S be the σ-algebra of all Lebesgue measurable subsets of D. Then f and μ are defined on S, and it is said that f is absolutely continuous relative to Lebesgue measure μ if and only if $f(M) = 0$ for every Lebesgue measurable set $M \subset S$ for which $\mu(M) = 0$.

Thus, it follows from Theorem 4.4 that a Lebesgue measurable mapping $f : D \rightarrow D'$ possesses property (N) and hence is absolutely continuous relative to Lebesgue measure μ.

The Generalized Jacobian of a Mapping

The theory of transformation or mappings in q-dimensional Euclidean space $(q > 1)$ is considerably more profound than that used in treatments given in the usual mathematical analysis or advanced calculus texts at the undergraduate level. The common change of variable theorems for both the

Lebesgue and the Riemann integrals place extremely stringent requirements upon the mapping. However, if the theory of Lebesgue measure is used, various general change of variable theorems can be proved without making any reference to the ordinary Jacobian, or even to differentiability. It is to this end that the "generalized Jacobian" is introduced. As anticipated it will replace the ordinary Jacobian and play its rôle as a type of pointwise (local) magnification element in the change of variable formulas which will be considered in Chapter VI.

Definition 4.6: Let $f : D \to D'$ be a mapping carrying $D \subset R_q$ into $D' \subset R'_q$. A cube K in R_q is defined as a closed cubic interval with center at $\bar{a} = (a_1, a_2, \ldots, a_q)$ and with edge length $s > 0$, where the q pairs of parallel sides are parallel to the q coordinate planes. More explicitly, a cube $K = K(\bar{a}:s)$ can be expressed as:

$$K = K(\bar{a}:\bar{s}) = \left\{ \bar{x} \;\middle|\; |x_j - a_j| \leq \frac{s}{2}, \quad j = 1, 2, \ldots, q. \right\}, \text{ with } s > 0.$$

Now let μ^* be q-dimensional Lebesgue outer measure, defined on all subsets of R_q. Then, for every $\bar{a} \in D$, define

$$\begin{aligned} V_f(\bar{a}:s) &= \frac{\mu^*[f(K \cap D)]}{\mu^*(K \cap D)}, \text{ providing } \mu^*(K \cap D) \neq 0 ; \\ &= +\infty \quad , \text{ when } \mu^*(K \cap D) = 0 \text{ and } \mu^*[f(K \cap D)] > 0 ; \\ &= 0 \quad , \text{ if } \mu^*(K \cap D) = \mu^*[f(K \cap D)] = 0 . \end{aligned}$$

Hence, $V_f(\bar{a}:s)$ is an extended-real-valued nonnegative function of s (defined for all $s > 0$). Now define the numbers $\overline{V}_f(\bar{a})$ and $\underline{V}_f(\bar{a})$ by

$$\overline{V}_f(\bar{a}) = \overline{\lim_{s \to 0^+}} V_f(\bar{a}:s)$$

and

$$\underline{V}_f(\overline{a}) = \underline{\lim}_{s \to 0^+} V_f(\overline{a}:s) .$$

If $\overline{V}_f(\overline{a}) = \underline{V}_f(\overline{a})$, it follows that $\lim_{s \to 0^+} V_f(\overline{a}:s)$ exists. This limit, when it exists, is denoted by the symbol $V_f(\overline{a})$ and is called the <u>generalized Jacobian</u> of the mapping f at the point $\overline{a} \in D$. As was mentioned previously, the generalized Jacobian has functional similarity with the ordinary Jacobian.

Although the definition above was made without the requirement of continuity of the mapping f, in the work which follows continuity of the mapping f will always be required; thus the above definition takes on a simpler form, especially in the case that $\overline{a}$ is an <u>interior</u> point of D. In this case, for sufficiently small s, $K = K(\overline{a}:s) \subset D$ and thus K and $f(K)$ are both closed sets. Now the definition of $V_f(\overline{a})$ takes on the simple form

$$V_f(\overline{a}) = \lim_{s \to 0^+} \frac{\mu[f(K)]}{\mu(K)}$$

whenever the limit exists.

<u>Definition 4.7</u>: Let $\overline{u} = (u_1, u_2, \ldots, u_q)$ be a fixed vector in R_q. The set of all vectors $\overline{x} \in R_q$ which satisfy the condition

$$\overline{x} \cdot \overline{u} = \sum_{j=1}^{q} x_j u_j = c ,$$

for a real constant c, is called a <u>hyperplane</u> in R_q.

<u>Definition 4.8</u>: Let $\overline{x}$ and $\overline{y}$ be vectors in R_q. An <u>affine</u> transformation

is a linear transformation of R_q into R'_q which has the form

$$\begin{Bmatrix} y_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q + c_1 \\ y_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2q}x_q + c_2 \\ \vdots \\ y_q = a_{q1}x_1 + a_{q2}x_2 + \dots + a_{qq}x_q + c_q \end{Bmatrix}, \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \vdots & \vdots & & \vdots \\ a_{q1} & a_{q2} & \dots & a_{qq} \end{bmatrix}$$

is a non-singular matrix of constant coefficients and $c = (c_1, c_2, \dots c_q)$ is a constant vector.

Definition 4.9: A q-dimensional parallelepiped K' in R'_q is a closed convex set of points which is the affine image of the q-dimensional cube $K = K(\overline{a};s)$ in R_q. Since the affine image of a hyperplane is also a hyperplane, it follows that the q-dimensional parallelepiped can be described as the closed set of all those points in R'_q which "lie" between q pairs of parallel hyperplanes. (Note: The hyperplanes are not in general parallel to the coordinate planes.)

Theorem 4.10: Let $f : D \rightarrow D'$ be a continuous mapping in R_q and let $\overline{a}$ be an interior point of D. Then a sufficient condition for the existence of the generalized Jacobian of f at $\overline{a}$ is that there exist two q-dimensional parallelepipeds K_o' and K_a', which are affine images of the cube $K(\overline{a};s)$ (for $0 < s < s_o$), such that

$$K_o' \subset f(K) \subset K_a'$$

and

$$\lim_{s\to 0^+} \frac{\mu(K_o')}{\mu(K)} = \lim_{s\to 0^+} \frac{\mu(K_a')}{\mu(K)} = L .$$

In the case when the limits in question exist and are equal, it is clear that $L = V_f(\overline{a})$.

<u>Proof</u>: For s sufficiently small ($0 < s < s_o$ say) the cube K is contained in D and thus $f(K) \subset D'$. Also

$$\mu^*(K_o') \leq \mu^*[f(K)] \leq \mu^*(K_a')$$

and since K_o' and K_a' are both Lebesgue measurable it follows that

$$(1) \qquad \frac{\mu(K_o')}{\mu(K)} \leq \frac{\mu^*[f(K)]}{\mu(K)} \leq \frac{\mu(K_a')}{\mu(K)} .$$

The desired results follow by taking the limit of both sides of inequality (1) as $s \to 0$. Thus the proof of the theorem is complete. ∎

CHAPTER V

DIFFERENTIABILITY OF THE LEBESGUE INTEGRAL IN q-DIMENSIONAL EUCLIDEAN SPACE

The theory of differentiation of the Lebesgue indefinite integral in R_q $(q > 1)$ is more involved than the corresponding theory for the one dimensional case. In this chapter the theory of the "regular derivative" of an arbitrary set function will be developed, and this concept will be utilized to prove Lebesgue's famous theorem concerning the differentiation of the indefinite integral. This proof can be found (using slightly different methods) in [9] (vol. II), [14], and in [18].

Regular Derivatives of Set Functions

Definition 5.1: Let M be any bounded Lebesgue measurable set in R_q. The cube $K(\overline{a}:s:M)$ is defined to be the smallest cube in R_q which has center at $\overline{a}$ and which contains the set M (such a cube will always exist).

Definition 5.2: Let $\overline{a}$ be point in R_q, r a positive number, and $X_1, X_2, \ldots, X_n, \ldots$ a sequence of bounded Lebesgue measurable subsets of R_q. The sequence $\left\{X_n\right\}$ is said to converge r-regularly to $\overline{a}$ provided: (1) for each positive integer n,

$$\frac{\mu(X_n)}{\mu\left[K(\overline{a}:s:X_n)\right]} \geq r ,$$

and (2) the edge $s > 0$ of $K(\bar{a}:s:X_n)$ tends to zero as $n \to +\infty$. Furthermore, the sequence $\{X_n\}$ is said to converge regularly to $\bar{a}$ if, for some $r > 0$, the sequence converges r-regularly to $\bar{a}$.

An example of a sequence of sets converging r-regularly is given by $\{K_n\}$, a contracting sequence of cubes with K_n defined by

$$K_n = \left\{ \bar{x} \quad |x_j - a_j| \leq \frac{s}{2n} \text{ , } j = 1,2,\ldots,q \right\}, \text{ for } s > 0 .$$

On the other hand, the intervals I_n in R_2 defined by

$$I_n = \left\{ \bar{x} \;|\bar{x} \quad R_q, \; |x_1| < \frac{1}{n} \; |x_2| < \frac{1}{n^2} \right\}$$

do not converge r-regularly to the origin for any positive r. Intuitively, requirement (1) of Definition 5.2 prevents, in some sense, the sets of a regularly converging sequence from being too flat. The positive number r, associated with a regularly convergent sequence of sets, is called the parameter of regularity and may be different for different sequences $\{X_n\}$.

Definition 5.3: Suppose that F is a generalized set function defined on the σ-algebra of Lebesgue measurable sets in R_q. The upper regular derivate and the lower regular derivate of F at point $\bar{x}$ in R_q are defined as follows:

$$\bar{D}\, F(\bar{x}) = \sup_{\{X_n\}} \ \overline{\lim_{n\to\infty}} \ \frac{F(X_n)}{\mu(X_n)} \quad ,$$

and

$$\underline{D}\, F(\bar{x}) = \inf_{\{X_n\}} \ \underline{\lim_{n\to\infty}} \ \frac{F(X_n)}{\mu(X_n)} \quad ,$$

where $\{X_n\}$ is any sequence of bounded Lebesgue measurable sets converging regularly to $\overline{x}$. In the definition of $\overline{D}F(\overline{x})$ and $\underline{D}F(\overline{x})$, the supremum and infimum are taken over all sequences $\{X_n\}$ converging regularly to $\overline{x}$. If $\overline{D}F(\overline{x})$ and $\underline{D}F(\overline{x})$ are equal and finite at $\overline{x}$, their common value is called the <u>regular derivative</u> of F at $\overline{x}$ and is denoted by $DF(\overline{x})$. Thus, in the case that $DF(\overline{x})$ exists,

$$\lim_{n\to\infty} \frac{F(X_n)}{\mu(X_n)} = DF(\overline{x})$$

for every sequence $\{X_n\}$ of bounded Lebesgue measurable sets which converges regularly to $\overline{x}$.

<u>Lebesgue's Theorem on Differentiating the Indefinite Integral</u>

The principal object of what follows is to show that if the function $f : R_q \to R^*$ is Lebesgue summable and $F(M)$ is the integral of f over M for every Lebesgue measurable set M, then $DF(\overline{x})$ exists and is equal to $f(\overline{x})$ almost everywhere in R_q. The following lemmas will be used in the proof.

<u>Lemma 5.4</u>: (a) Let $u : R_q \to R^*$ be a lower semicontinuous Lebesgue summable function which is bounded below and let $U(M) = \int_M u(\overline{x})\, d\mu$ for every Lebesgue measurable set M. Then it follows that $\underline{D}U(\overline{x}) \geq u(\overline{x})$ for all $\overline{x} \in R_q$.

(b) Let $\ell : R_q \to R^*$ be an upper semicontinuous Lebesgue summable function which is bounded above and let

$$L(M) = \int_M \ell(\overline{x})\, d\mu$$

for every Lebesgue measurable set M. Then it follows that $\overline{D}L(x) \leq \ell(\overline{x})$

for all $\bar{x} \in R_q$.

Proof: (a) Let $\bar{x}_o$ be an arbitrary point in R_q and let h be less than $u(\bar{x}_o)$. Since $u(\bar{x})$ is L.S.C., it follows that there exists a neighborhood $N(\bar{x})$ in which $u(\bar{x})$ is greater than h (cf. Theorem 2, Appendix). Let $\{X_n\}$ be a sequence of bounded Lebesgue measurable sets tending regularly to $\bar{x}_o$. Ultimately the set X_n is contained in $N(\bar{x}_o)$ (for all n sufficiently large), so that

$$\frac{U(X_n)}{\mu(X_n)} = \frac{\int_{X_n} u(\bar{x})\, d\mu}{\mu(X_n)} \geq \frac{h\mu(X_n)}{\mu(X_n)} = h$$

and thus

$$\underline{\lim}_{n\to\infty} \frac{U(X_n)}{\mu(X_n)} \geq h .$$

Therefore $\underline{D}U(\bar{x}_o) \geq h$. Since h can be any number less than $u(\bar{x}_o)$, it follows that $\underline{D}U(\bar{x}_o) \geq u(\bar{x}_o)$. But, since $\bar{x}_o \in R_q$ is arbitrary, the proof of Lemma 5.4 (a) is complete. Part (b) is proved by a similar argument.

The next lemma is a weakened form of the Vitali Covering Theorem.

Lemma 5.5: Let Γ be a family of non-degenerate cubes (each closed in R_q) and let G be the union of the interiors of the cubes K of Γ. Then, for every number h less than $\frac{\mu(G)}{5^q}$, there exists a finite disjoint collection of cubes $K_1, K_2, \ldots, K_p$ such that

$$\sum_{j=1}^{j=p} \mu(K_j) > h.$$

Proof: A proof of the above lemma may be found in McShane and Botts (cf. [14], pp. 184-5); due to its importance it is given below.

Let b_o be the supremum of the edge lengths of the cubes K of Γ. If $b_o = +\infty$, the desired result follows with $p = 1$. If b_o is finite, a cube K_1 of Γ is chosen with edge length greater than $\frac{b_o}{2}$. Inductively, if for a positive integer n, the numbers $b_o, \ldots, b_{n-1}$ and the cubes $K_1, \ldots, K_n$ have been defined, then define b_n to be the supremum of the edges of all members K of Γ disjoint from $K_1, \ldots, K_n$ if there are such cubes, and to be zero otherwise; in the former case choose K_{n+1} to be a member of Γ disjoint from $K_1, \ldots, K_n$ and with edge greater than $\frac{b_n}{2}$, while in the latter case $(b_n = 0)$ choose K_{n+1} to be the empty set. Then it follows that $b_o \geq b_1 \geq \ldots$. If the b_n do not converge to zero, the measures of the K_n remain above a positive lower bound, so that the infinite series $\mu(K_1) + \mu(K_2) + \ldots$ has sum $+\infty$, and a finite partial sum $\mu(K_1) + \ldots + \mu(K_p)$ exceeds h. There remains the case in which b_n tends to zero as n increases.

For each n for which K_n is nonempty, let K_n^* be the closed cube with the same center as K_n and with edge length five times as great; and for each other n, let K_n^* be the empty set. Let $\bar{x}$ be any point of G; it is interior to a cube K' of Γ. The edge length of K' is positive, but not greater than b_o, so the least integer m, such that $b_m <$ edge length of K', is positive. Then by the definition of b_m, one of the intervals $K_1, \ldots, K_m$ (say K_j) must have a point in common with K'. Since m is the least integer for which $b_m <$ edge K', it follows that $b_{j-1} \geq$ edge length of K', so the edge length of $K_j > \frac{\text{edge length of } K'}{2}$. Therefore K' lies entirely in the interval

K_j^*, and $\overline{x} \in K_j^*$. Thus it has been shown that $G \subset \bigcup_{j=1}^{\infty} K_j^*$. Hence

$$\sum_{j=1}^{\infty} \mu(K_j^*) \geq \mu(G) ,$$

so

$$\sum_{j=1}^{\infty} \mu(K_j) \geq \frac{\mu(G)}{5^q} > h ,$$

and for some integer p it is true that

$$\sum_{j=1}^{p} \mu(K_j) > h .$$

This completes the proof of the lemma. ∎

The principal theorem of this chapter will now be proved.

Theorem 5.6: Let f be a real-valued function which is defined and Lebesgue summable over R_q. Furthermore, for each Lebesgue measurable set M, define the set function F by

$$F(M) = \int_M f(\overline{x}) \, d\mu .$$

Then, for almost all points $\overline{x} \in R_q$, the regular derivative $DF(\overline{x})$ exists and is equal to $f(\overline{x})$.

Proof: Let N be the set of all $\overline{x} \in R_q$ at which the regular derivative of $F(\overline{x})$ either fails to exist or exists and is different from $f(\overline{x})$. If $\overline{x} \in N$, there exists a sequence $\{X_n'\}$ converging r-regularly to $\overline{x}$ (for some $r > 0$) such that it is false that

$$\lim_{n\to\infty} \frac{F(X_n')}{\mu(X_n')} = f(\overline{x}) .$$

Thus, there exists a positive integer k and a subsequence $\{X_n\}$ of $\{X_n'\}$ such that

$$\left| \frac{F(X_n)}{\mu(X_n)} - f(\overline{x}) \right| \geq \frac{1}{k} \quad (n = 1,2,\dots) .$$

Choose a positive integer j such that $j \geq k$ and $\frac{1}{j} < r$. Then the sequence $\{X_n\}$ converges $\frac{1}{j}$-regularly to $\overline{x}$ while at the same time

$$\left| \frac{F(X_n)}{\mu(X_n)} - f(\overline{x}) \right| \geq \frac{1}{j} \quad (n = 1,2,\dots) .$$

For each positive integer j, let N_j be the set of all $\overline{x} \in R_q$ such that there exists a sequence $\{X_n\}$ of bounded Lebesgue measurable sets converging $\frac{1}{j}$-regularly to $\overline{x}$ and having

$$\left| \frac{F(X_n)}{\mu(X_n)} - f(\overline{x}) \right| \geq \frac{1}{j} \quad (n = 1,2,\dots).$$

Clearly N is the union of all the N_j, so the theorem will be proved if it is established that $\mu(N_j) = 0$ for each $j = 1,2,\dots$. Assume that there exists a positive integer j such that N_j fails to have Lebesgue measure zero. In this way a contradiction will be obtained.

If N_j were to be contained in an open set of arbitrarily small measure, N_j would have measure zero, contrary to the above assumption; hence, there is a positive number $\epsilon > 0$ such that for every open set G containing N_j, $\mu(G) > \epsilon$. Now, since f is Lebesgue summable over

R_q, by Theorem 2.6 (the Theorem of Vitali-Carathéodory) there exist two functions u and ℓ which have the following properties:

(i) u and ℓ are Lebesgue summable over R_q.

(ii) u is L.S.C. and is bounded below and ℓ is U.S.C. and is bounded above.

(iii) $\ell(\overline{x}) \leq f(\overline{x}) \leq u(\overline{x})$ for every $\overline{x} \in R_q$.

(iv) $\int_{R_q} (u - \ell)\, d\mu < \frac{\epsilon}{2 \cdot 5^q j^2}$.

For each Lebesgue measurable set M define $U(M)$ and $L(M)$ to be the integrals over M of u and ℓ respectively, and denote by Γ the family of all nondegenerate cubes K such that K contains a Lebesgue measurable set $X(K)$ with

$$\frac{[U(X(K)) - L(X(K))]}{\mu(K)} \geq \frac{1}{2j^2} .$$

It will be shown that the union G of the interiors of the cubes K of Γ contains the set N_j. Suppose that $\overline{x}$ belongs to N_j; then there is a sequence $\{X_n\}$ of Lebesgue measurable sets converging $\frac{1}{j}$-regularly to $\overline{x}$ and having

$$\left| \frac{F(X_n)}{\mu(X_n)} - f(\overline{x}) \right| \geq \frac{1}{j} \quad \text{for } n = 1,2,\ldots .$$

For each sequence $\{X_n\}$ converging $\frac{1}{j}$-regularly to $\overline{x}$, a subsequence $\{X_{n_k}\}$ $(k = 1,2,\ldots)$ can be extracted such that $\frac{F(X_{n_k})}{\mu(X_{n_k})}$ converges $\frac{1}{j}$- regularly to $\overline{\lim_{n\to\infty}} \frac{F(X_n)}{\mu(X_n)} = c$. Thus, without loss of generality, it can be assumed that the entire sequence $\dfrac{F(X_n)}{\mu(X_n)}$ converges to c. Now since $u(\overline{x}) \geq f(\overline{x})$, it follows that

$$\underline{\lim}_{n\to\infty} \frac{U(X_n)}{\mu(X_n)} \geq \underline{\lim}_{n\to\infty} \frac{F(X_n)}{\mu(X_n)} = \lim_{n\to\infty} \frac{F(X_n)}{\mu(X_n)} = c \text{ ,}$$

and by Lemma 5.4 (a) ,

$$\underline{\lim}_{n\to\infty} \frac{U(X_n)}{\mu(X_n)} \geq DU(\overline{x}) \geq u(\overline{x}) \geq f(\overline{x}) \text{ .}$$

Likewise, by Lemma 5.4 (b),

$$\overline{\lim}_{n\to\infty} \frac{L(X_n)}{\mu(X_n)} \leq c \text{ ,}$$

and

$$\overline{\lim}_{n\to\infty} \frac{L(X_n)}{\mu(X_n)} \leq f(\overline{x}) \text{ .}$$

Therefore both $f(\overline{x})$ and c are between $\overline{\lim}_{n\to\infty} \dfrac{L(X_n)}{\mu(X_n)}$ and $\underline{\lim}_{n\to\infty} \dfrac{U(X_n)}{\mu(X_n)}$, so these must differ by $|f(\overline{x}) - c|$ or more. Hence

$$\underline{\lim}_{n\to\infty} \frac{[U(X_n) - L(X_n)]}{\mu(X_n)} \geq \underline{\lim}_{n\to\infty} \frac{U(X_n)}{\mu(X_n)} - \overline{\lim}_{n\to\infty} \frac{L(X_n)}{\mu(X_n)}$$

$$\geq |f(\overline{x}) - c| \text{ .}$$

But, since $\overline{x} \in N_j$, it follows that $|f(\overline{x}) - c| \geq \frac{1}{j}$, and thus, there exists an integer n such that

$$\frac{[U(X_n) - L(X_n)]}{\mu(X_n)} > \frac{1}{2j} \text{ .}$$

Now let $K = K(\overline{x};s;X_n)$. By the regularity property of the sequence $\{X_n\}$, it follows that

$$\frac{\mu(X_n)}{\mu(K)} \geq \frac{1}{j}$$

so that

$$\frac{U(X_n) - L(X_n)}{\mu(K)} \geq \frac{1}{2j^2} .$$

Thus $K \in \Gamma$ and $\overline{x}$ is an interior point of K. Hence G (defined earlier) is an open set containing N_j, and thus, in view of the discussion at the beginning of the proof, $\mu(G) > \epsilon$.

By Lemma 5.5 there exists a finite subfamily $K_1, K_2, \ldots, K_p$ of pairwise disjoint cubes of the family Γ such that

$$\sum_{i=1}^{i=p} \mu(K_i) > \frac{\epsilon}{5^q} .$$

The corresponding Lebesgue measurable subsets $X(K_1)$, $X(K_2)$, $\ldots$, $X(K_p)$ are pairwise disjoint and

$$\sum_{i=1}^{i=p} \left[U(X(K_i)) - L(X(K_i))\right] \geq \frac{\epsilon}{5^q 2j^2} .$$

But the left member of the last inequality is just the integral of $u - \ell$ over $\bigcup_{i=1}^{p} X(K_i)$. Hence, if the integral (which is non-negative) of $u - \ell$ over the rest of R_q is added to it, then

$$\int_{R_q} (u-\ell)\, d\mu \geq \frac{\epsilon}{2\cdot 5^q j^2} \, .$$

This contradicts property (iv) of the functions u and ℓ. Thus the theorem is established. ∎

CHAPTER VI

GENERAL CHANGE OF VARIABLE THEOREMS FOR THE LEBESGUE INTEGRAL IN q-DIMENSIONAL EUCLIDEAN SPACE

In this chapter two change of variable theorems will be proved which exhibit the effect of a change of variables in a Lebesgue integral. These theorems will be proved by making use of the differentiation theory which was developed in Chapter V. The outstanding feature of these theorems is that the transformation (mapping) f will not be required to be differentiable in its domain of definition; accordingly, the generalized Jacobian V_f, which was defined and discussed in Chapter IV, will replace the ordinary Jacobian in the transformation formula. Although this non-inductive method of proof is contingent upon the differentiation theory developed in Chapter V and upon the theory of the generalized Jacobian of Chapter IV, the proofs given here are, as a whole, much simpler in principle than those involved in the inductive approaches which are commonly found in the literature (cf. [14], [19], or [21]). Since Lebesgue measure μ is always the measure in question, the phrase "almost everywhere relative to Lebesgue measure μ" will be replaced by the letters "a.e."

Lemma 6.1: Let D be a Lebesgue measurable set in R_q, and let C_D be the characteristic function of the set D. For every Lebesgue measurable set $M \subset R_q$ define the function g by

$$g(M) = \int_M C_D(\overline{x})\, d\mu .$$

Then it follows that $Dg(\overline{x}) = 1$ a.e. in D.

Proof: For each positive integer k define the set W_k in R_q by

$$W_k = \left\{ \overline{x} \;\middle|\; |x_j| \leq k, \quad j = 1,2,\dots,q \right\}.$$

Also, for every Lebesgue measurable set $M \subset R_q$, define the function g_k by

$$g_k(M) = \int_M C_{D \cap W_k}(\overline{x})\, d\mu ,$$

where $C_{D \cap W_k}$ is the characteristic function of the set $D \cap W_k (k = 1,2,\dots)$. By Theorem 5.6 it follows that

$$Dg_k(\overline{x}) = C_{D \cap W_k}(\overline{x}) \quad \text{a.e. in } R_q .$$

Thus

$$Dg_k(\overline{x}) = 1 \quad \text{a.e. in } D \cap W_k .$$

Now, for each positive integer k, define the set S_k by

$$S_k = \left\{ \overline{x} \mid \text{all } \overline{x} \in D \text{ such that } Dg_k(x) \neq 1 \text{ or } Dg_k(\overline{x}) \text{ does not exist} \right\}.$$

Clearly $\mu(S_k) = 0$ for each $k = 1,2,\dots$. Thus

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) \leq \sum_{k=1}^{k=\infty} \mu(S_k) = 0 .$$

Let $\overline{x}_o$ be any point in the set $D - \bigcup_{k=1}^{\infty} S_k$. Then, for k sufficiently large $(k \geq k_1$ say), the point $\overline{x}_o$ is interior to W_k. Now let $\left\{K_n(\overline{x}_o : \frac{1}{n})\right\}$ be a sequence of cubes converging regularly to $\overline{x}_o$. It follows that, for n sufficiently large ($n \geq n_1$ and $k \geq k_1$ say), $K_n \subset W_k$. Hence

$$g_k(K_n) = \mu(K_n \cap D \cap W_k) = \mu(K_n \cap D) = g(K_n) \ ,$$

and thus

$$Dg(\overline{x}_o) = Dg_k(\overline{x}_o) = 1 \ .$$

But $\overline{x}_o$ is an arbitrary point in $D - \bigcup_{k=1}^{\infty} S_k$; this implies that $Dg(\overline{x}) = 1$ a.e. in D. Thus the proof is complete. ∎

Theorem 6.2: Let $f : D \to D'$ be a continuous measurable mapping which is one-to-one in the large on D. Let $D \subset R_q$ be Lebesgue measurable and let f^{-1} be the (unique) inverse of the mapping f. Then the following statements are true:

(i) The mapping f possesses (a.e. in D) a finite q-dimensional generalized Jacobian $V_f(\overline{x})$ which agrees (a.e. in D) with a nonnegative Lebesgue integrable function $T_f(\overline{x})$ defined on D.

(ii) The measure $\mu(M')$ of the image $M' = f(M)$ of every Lebesgue measurable subset $M \subset D$ can be calculated using the formula

$$(1) \qquad \mu(M') = \int_M T_f(\overline{x}) \, d\mu \ ,$$

where μ is q-dimensional Lebesgue measure.

Proof: (i) and (ii). Define the function F by

$$F(M) = \mu\left[f(D \cap M)\right]$$

for all Lebesgue measurable sets $M \subset R_q$. Note that if $M \subset D$, $F(M) = \mu[f(M)]$.

Let A and B be two disjoint Lebesgue measurable sets in R_q. Since the mapping f is one-to-one, $f(A) \cap f(B) = \Phi$, where Φ is the empty set. Now let $M_1, M_2, \ldots, M_n, \ldots$ be a sequence of pairwise disjoint Lebesgue measurable sets in R_q. Then, since f is one-to-one and Lebesgue measurable, it follows that

$$F\left[\bigcup_{n=1}^{\infty} M_n\right] = \mu\left[f(D \cap \bigcup_{n=1}^{\infty} M_n)\right] = \mu\left[f\left(\bigcup_{n=1}^{\infty} (D \cap M_n)\right)\right]$$

$$= \mu\left[\bigcup_{n=1}^{\infty} f(D \cap M_n)\right] = \sum_{n=1}^{\infty} \mu[f(D \cap M_n)] = \sum_{n=1}^{\infty} F(M_n) \ .$$

Hence F is countably additive on (R_q, S, μ), where the triple (R_q, S, μ) denotes the measure space with S as the σ-algebra of Lebesgue measurable subsets of R_q.

Since f is a Lebesgue measurable mapping, f is A.C. relative to Lebesgue measure μ. Thus, if N is a null set in R_q,

$$F(N) = \mu[f(D \cap N)] = 0 \ .$$

Therefore F is A.C. relative to Lebesgue measure μ.

The set function F is nonnegative, completely additive, and A.C. relative to Lebesgue measure μ. It will now be shown that F is a totally σ-finite measure on (R_q, S, μ). Moreover, since

$$F(R_q - D) = \mu[f(\Phi)] = 0 \ ,$$

it need only be shown that F is totally σ-finite on D. Since D' is in R_q', there exist compact sets $C_1', C_2', \ldots, C_j', \ldots$ such that $D' \subset \bigcup_{j=1}^{\infty} C_j' = R_q'$ with $\mu(C_j') < +\infty$ for each $j = 1,2,\ldots$. Define the set $D_j' = D' \cap C_j'$ $(j = 1,2,\ldots)$. Then D_j' is contained in D' and is closed in D' $(j = 1,2,\ldots)$. Furthermore, $D' = \bigcup_{j=1}^{\infty} D_j'$ and thus

$$D = f^{-1}(D') = f^{-1}\left[\bigcup_{j=1}^{\infty} D_j'\right] = \bigcup_{j=1}^{\infty} f^{-1}(D_j') = \bigcup_{j=1}^{\infty} D_j ,$$

where $D_j = f^{-1}(D_j')$ $(j = 1,2,\ldots)$. Since the mapping f is continuous and one-to-one on D, it follows that f^{-1} is continuous on D'. Thus D_j is closed relative to D. It now follows that

$$F(D_j) = \mu\left[f(D \cap D_j)\right] = \mu\left[f(D_j)\right] = \mu(D_j') < +\infty$$

for each $j = 1,2,\ldots$. Thus F is totally σ-finite on (R_q, S, μ).

Therefore the hypotheses of the Radon-Nikodym Theorem (Theorem 5, Appendix) are satisfied; hence, the existence of a real-valued function T_f (defined on R_q) is deduced such that

$$F(M) = \int_M T_f(\bar{x})\, d\mu \quad \text{for every Lebesgue measurable set } M \subset R_q.$$

Moreover, T_f is unique (modulo μ). In addition, since $F(M)$ is non-negative for every Lebesgue measurable set M, it follows that $T_f(\bar{x}) \geq 0$ a.e. in D. Also $T_f(\bar{x}) = 0$ a.e. in $R_q - D$. Thus, without loss of generality, it will be required that the function $T_f(\bar{x}) \geq 0$ for every $\bar{x} \in D$ and that $T_f(\bar{x})$ vanish outside of D.

Now, since the function T_f is not in general Lebesgue summable over R_q, the total σ-finiteness of F on (R_q, S, μ) will be used to prove that the regular derivative $DF(\bar{x})$ exists and equals $T_f(\bar{x})$

a.e. in D.

Express D as a countable disjoint union $D = \bigcup_{j=1}^{\infty} D_j$ of Lebesgue measurable sets such that $F(D_j) < +\infty$ for each $j = 1,2,\ldots$. Since each D_j is Lebesgue measurable, given $\epsilon_j > 0$, there exists an open set G_j such that $D_j \subset G_j$ and

$$F(G_j) < \epsilon_j + F(D_j) .$$

In addition,

$$D = \bigcup_{j=1}^{\infty} D_j \subset \bigcup_{j=1}^{\infty} G_j .$$

Moreover, the function T_f is Lebesgue summable over G_j since

$$F(G_j) = \int_{G_j} T_f(\overline{x})\, d\mu < \epsilon_j + F(D_j) < +\infty \ (j = 1,2,\ldots).$$

Define the functions T_f^j and F^j by:

$$\begin{aligned} T_f^j(\overline{x}) &= T_f(\overline{x}) \ , \quad x \in G_j \ ; \\ &= 0 \qquad , \quad \overline{x} \in R_q - G_j ; \end{aligned}$$

and

$$F^j(M) = \int_M T_f^j(\overline{x})\, d\mu \quad \text{for every Lebesgue measurable set } M \subset R_q .$$

Clearly, each function F^j is Lebesgue summable over R_q since

$$F^j(R_q) = \int_{R_q} T_f^j(\overline{x})\, d\mu = \int_{G_j} T_f(\overline{x})\, d\mu = F(G_j) < +\infty .$$

Thus, according to Theorem 5.6, the regular derivative $DF^j(\overline{x})$ of F^j

exists and equals $T_f^j(\overline{x})$ a.e. in R_q. In particular,

$$DF^j(\overline{x}) = T_f^j(\overline{x}) \text{ a.e. in } G_j \text{ for each } j = 1,2,\ldots .$$

And, since G_j is open, it follows that

$$DF(\overline{x}) = DF^j(\overline{x}) \text{ a.e. on } G_j .$$

This implies that $DF(\overline{x}) = T_f(\overline{x})$ a.e. on G_j since $T_f^j(\overline{x}) = T_f(\overline{x})$ for every $\overline{x} \in G_j$ $(j = 1,2,\ldots)$. Thus $DF(\overline{x}) = T_f(\overline{x})$ a.e. on D since $D \subset \bigcup_{j=1}^{\infty} G_j$.

It follows from the definition of the regular derivative that, for every sequence $\{X_n\}$ converging regularly to $\overline{x}$,

$$\lim_{n\to\infty} \frac{F(X_n)}{\mu(X_n)} = DF(\overline{x}) \text{ a.e. in } D .$$

In particular, define a sequence of cubes $\{K_n\}$ by

$$K_n = \left\{ \overline{y} \;\middle|\; |x_i - y_i| \le \frac{s}{2n} ,\ i = 1,2,\ldots,q \right\}, \text{ for } s > 0 .$$

It is obvious that the sequence $\{K_n\}$ converges regularly to $\overline{x}$. Hence, for almost all points $\overline{x} \in D$,

$$DF(\overline{x}) = \lim_{n\to\infty} \frac{F(K_n)}{\mu(K_n)} = \lim_{s\to 0} \frac{\mu[f(K_n \cap D)]}{\mu(K_n)} \text{ for } n > 0 .$$

But

$$V_f(\overline{x}) = \lim_{s\to 0} \frac{\mu[f(K_n \cap D)]}{\mu(K_n \cap D)} = \lim_{s\to 0} \frac{\mu f(K_n \cap D)}{\mu(K_n)} \cdot \frac{\mu(K_n)}{\mu(K_n \cap D)} ,$$

and, since

$$\mu(K_n \cap D) = \int_{K_n} K_D(\overline{x})\, d\mu ,$$

Lemma 6.1 implies that

$$\lim_{n\to\infty} \frac{\mu(K_n \cap D)}{\mu(K_n)} = 1 \quad \text{a.e. in } D ;$$

thus

$$\lim_{n\to\infty} \frac{\mu(K_n)}{\mu(K_n \cap D)} = 1 \quad \text{a.e. in } D .$$

Hence

$$\lim_{s\to 0} \frac{\mu(K_n)}{\mu(K_n \cap D)} = 1 \quad \text{a.e. in } D$$

which implies that

$$V_f(\overline{x}) = DF(\overline{x}) \quad \text{a.e. in } D .$$

This also yields $V_f(\overline{x}) = T_f(\overline{x})$ a.e. in D; thus Equation (1) takes on the following suggestive form

$$(2) \qquad \mu(M') = \int_M V_f(\overline{x})\, d\mu$$

for every Lebesgue measurable set $M \subset D$. In Equation (2) it is understood that $V_f(\overline{x})$ is set equal to zero at values of $\overline{x}$ for which $V_f(\overline{x}) \neq T_f(\overline{x})$, or for which $V_f(\overline{x})$ does not exist. This completes the proof of the theorem. ∎

Theorem 6.2 will now be used to prove the following important change of variable formula.

Theorem 6.3: (Change of Variable Theorem for the Lebesgue Integral in R_q) Let $f : D \to D'$ be a continuous Lebesgue measurable mapping which is one-to-one in the large on the Lebesgue measurable set $D \subset R_q$. Further let f^{-1} be the (unique) inverse of f, defined on D'. Also let the mapping f possess a unique (modulo μ) q-dimensional generalized Jacobian V_f a.e. on D which agrees a.e. on D with a non-negative integrable function T_f. Then, for every Lebesgue measurable function $\overline{y} = f(\overline{x})$ which is defined on a Lebesgue measurable subset $M \subset D$, it is true that:

(1) The function $g(\overline{y}) = h[f^{-1}(\overline{y})]$ is Lebesgue measurable on the measurable image $M' = f(M)$ of M; and, if the integrals exist,

$$\int_M g[f(\overline{x})]\ T_f(\overline{x})\ d\mu = \int_{M'} g(\overline{y})\ d\mu\ , \tag{2}$$

where the product $h(\overline{x})T_f(\overline{x})$ in the left integral of (2) is set equal to zero on the set of points at which either $h(\overline{x})$ or $T_f(\overline{x})$ is zero.

Before considering the proof of this theorem, it is interesting to note that even though change of variable theorems for the Lebesgue integral are known, correct proofs of such theorems are difficult to find. Furthermore, to the knowledge of this author, a theorem comparable to Theorem 6.3 can only be found in the literature in one place (cf. [9], Vol. III, pp. 143-144). On the other hand, inductive proofs of change of variable theorems for the Lebesgue integral in R_q may be found in [14], [19], and [21]. All of these theorems, however, are quite specialized and require continuous differentiability of the mapping f.

<u>Proof</u>: (1) Set $g(\overline{y}) = h[f^{-1}(\overline{y})] = h(\overline{x})$. Since f is a Lebesgue measurable mapping, $M' = f(M)$ is Lebesgue measurable. Furthermore, since

h is Lebesgue measurable on M, it follows that the set

$$S = \left\{\overline{x} \mid \overline{x} \in M,\ h(\overline{x}) < c,\ c \text{ a real number}\right\}$$

is Lebesgue measurable. Thus the set $f(S)$ defined by

$$\begin{aligned} f(S) &= \left\{ f(\overline{x}) \mid \overline{x} \in M,\ h(\overline{x}) < c,\ c \text{ a real number}\right\} \\ &= \left\{ \overline{y} \mid \overline{y} \in M',\ g(\overline{y}) < c,\ c \text{ a real number} \right\} \end{aligned}$$

is also Lebesgue measurable; hence, g is Lebesgue measurable on M'. This completes the proof of assertion (1).

The proof of part (2) will now be considered. First of all assume that $h(\overline{x}) = g(\overline{y}) = c$, where c is a real constant. By Theorem 6.2

$$\int_M T_f(\overline{x})\, d\mu = \mu(M').$$

But

$$\mu(M') = \int_{M'} d\mu \ .$$

Thus, (2) assumes the desired form

$$\int_M c\ T_f(\overline{x})\, d\mu = \int_{M'} c\ d\mu \ .$$

This proves (2) in the case that $h(\overline{x}) = g(\overline{y}) = c$, where c is a constant.

Suppose now that h is a nonnegative, finite valued, Lebesgue measurable simple function defined on M; likewise, g will be a nonnegative Lebesgue measurable simple function defined on M'. Then there exist measurable sets $D_1, D_2, \ldots, D_n$ such that $D_j \cap D_k = \Phi$, for $j \neq k$ and $M = \bigcup_{k=1}^{n} D_k$. Furthermore

$$M' = f(M) = \bigcup_{k=1}^{n} f(D_k)$$

where $\bigcup_{k=1}^{n} f(D_k)$ is the corresponding measurable decomposition of M'. Hence, using the definition of a Lebesgue measurable simple function, it follows that

$$h(\overline{x}) = c_k \, , \; \overline{x} \in D_k$$

and

$$g(\overline{y}) = c_k, \; \overline{y} \in f(D_k) \quad \text{for each} \quad k = 1,2,\ldots,n \, .$$

Since the function T_f is nonnegative and Lebesgue measurable, it follows that $h\ T_f$ is Lebesgue integrable over M. Likewise, the nonnegative Lebesgue measurable simple function g is Lebesgue integrable over M'. Further, for $\overline{x} \in D_k$ and $\overline{y} \in f(D_k)$, $h(\overline{x}) = g(\overline{y}) = c_k$; thus, by the previous part of the argument, (2) holds for each set D_k $(k = 1,2,\ldots,n)$, and

$$\int_{D_k} h(\overline{x})\ T_f(\overline{x})\ d\mu = \int_{f(D_k)} g(\overline{y})\ d\mu \, .$$

Hence

$$\sum_{k=1}^{n} \int_{D_k} h(\overline{x})\ T_f(\overline{x})\ d\mu = \sum_{k=1}^{n} \int_{f(D_k)} g(\overline{y})\ d\mu$$

or

$$\int_{M} h(\overline{x}) \cdot T_f(\overline{x})\ d\mu = \int_{f(M)} g(\overline{y})\ d\mu \, .$$

Therefore the theorem holds for nonnegative Lebesgue measurable simple functions.

Suppose now that $h = g$ is nonnegative and Lebesgue measurable, but otherwise arbitrary. Then each of the integrals in (2) exist; equality is proved using the following argument. Since $h = g$ is nonnegative and Lebesgue measurable, there exists a nondecreasing sequence of nonnegative measurable simple functions $h_n = g_n$ for $n = 1,2,\ldots$ such that

$$\lim_{n\to\infty} h_n(\overline{x}) = h(\overline{x})$$

and

$$\lim_{n\to\infty} g_n(\overline{y}) = g(\overline{y})$$

for each $\overline{x} \in M$ and for each $\overline{y} \in M'$. Now, since the function T_f is nonnegative, the functions $h_n T_f$ form a nondecreasing sequence of nonnegative Lebesgue integrable functions (likewise for the sequence $\{g_n\}$). From the preceding part of the argument it follows that

$$\int_M h_n(\overline{x})\ T_f(\overline{x})\ d\mu = \int_{M'} g_n(\overline{y})\ d\mu \quad \text{for each} \quad n = 1,2,\ldots .$$

Thus, if the Monotone Convergence Theorem is applied, the desired equality

$$\int_M h(\overline{x})\ T_f(\overline{x})\ d\mu = \int_{M'} g(\overline{y})\ d\mu \tag{3}$$

is obtained.

The general case will now be considered. Suppose that the functions $h\,T_f$ and g are Lebesgue integrable over M and M' respectively. Let each of the functions h and g be decomposed into its positive and

and negative parts. Thus, let

$$h(\overline{x}) = h^+(\overline{x}) - h^-(\overline{x}) \quad \text{for} \quad \overline{x} \in M \ ,$$

and

$$g(\overline{y}) = g^+(\overline{y}) - g^-(\overline{y}) \quad \text{for} \quad \overline{y} \in M' .$$

Now, from the part of the argument for nonnegative Lebesgue measurable functions, it follows that

$$\int_M h^+(\overline{x})\ T_f(\overline{x})\ d\mu = \int_{M'} g^+(\overline{y})\ d\mu \tag{4}$$

and

$$\int_M h^-(\overline{x})\ T_f(\overline{x})\ d\mu = \int_{M'} g^-(\overline{y})\ d\mu \ . \tag{5}$$

Either the integrals in (4) or the integrals in (5) are finite. In either case, by subtraction of (5) from (4) assertion (2) is established. Thus

$$\int_M h(\overline{x})\ T_f(\overline{x})\ d\mu = \int_{M'} g(\overline{y})\ d\mu \ .$$

It is obvious from the above argument that integrability of $h\ T_f$ over M implies that of g over M' and vice versa. In fact, if $h\ T_f$ is Lebesgue summable over M, then so is g over M'. This completes the proof of the theorem. ∎

CHAPTER VII

SPECIAL CLASSES OF TRANSFORMATIONS

Although the change of variable theorem of Chapter VI (Theorem 6.3) is indispensable in applications involving transformations for which differentiability is not postulated, its value can not be fully appreciated until it is shown that the usual integral transformation theorems are special cases of it. To this end the final two chapters of the text are dedicated. Indeed, in popular applications the mapping considered is usually "well-behaved" so that its differentiability properties may be considered. Noteworthy indeed is the fact that at points of differentiability of a mapping the ordinary Jacobian is defined and assumes its rôle as the local magnification element in the transformation formula. Throughout the remainder of the text the ordinary Jacobian, whenever it exists, will be related in its essence and in function to the generalized Jacobian. Moreover, mappings of bounded expansion, which are of intermediate generality, will be defined and linked to the general transformation theory for the Lebesgue integral. In the final chapter certain specialized change of variable formulas will be proved in which the results obtained here will be utilized.

The Ordinary Jacobian

Definition 7.1: Let the mapping $f : D \rightarrow D'$ be differentiable at a point $\overline{x} = (x_1, x_2, \ldots, x_q) \in D$, where $D \subset R_q$ and $D' \subset R'_q$. The (ordinary) Jacobian of the mapping $f = (f_1, f_2, \ldots, f_q)$ is defined to be the real-valued function J_f whose values are given by the determinant

$$J_f(\overline{x}) = \begin{vmatrix} D_1f_1(\overline{x}) & D_2f_1(\overline{x}) & \cdots & D_qf_1(\overline{x}) \\ \vdots & \vdots & & \vdots \\ D_1f_q(\overline{x}) & D_2f_q(\overline{x}) & \cdots & D_qf_q(\overline{x}) \end{vmatrix}$$

at those points $\overline{x} \in D$ where all partials $D_jf_i(\overline{x})$ exist $(i,j=1,2,\ldots,q)$.

If a mapping $f = (f_1,f_2,\ldots f_q)$ is differentiable at an interior point $\overline{a} \in D$, this implies that each f_j is differentiable at $\overline{a} \in D$. Hence, from the definition of the differential (cf. [1] or [14]), it is required that

$$\frac{f_j(\overline{x}) - f_j(\overline{a}) - \sum_{k=1}^{q} D_kf_j(\overline{a})\ (x_k - a_k)}{\rho(\overline{x},\overline{a})} \to 0 \quad \text{as} \quad \overline{x} \to \overline{a} \ (j=1,2,\ldots,q).$$

Thus the notation

$$f_j(\overline{x}) - f_j(\overline{a}) - \sum_{k=1}^{q} D_kf_j(\overline{a})(x_k - a_k) = o(\rho(\overline{x},\overline{a})) \quad (j=1,2,\ldots,q)$$

will be used to avoid a conglomeration of ϵ's which would naturally occur where differentials are to be manipulated. Here $o(\cdot)$ denotes a term, say $h_j(\overline{x})$, such that

$$\frac{h_j(\overline{x})}{\rho(\overline{x},\overline{a})} \to 0 \quad \text{as} \quad \overline{x} \to \overline{a} .$$

Essentially $h_j(\overline{x})$ tends to zero more <u>rapidly</u> than $\rho(\overline{x},\overline{a})$.

The purpose of the following theorem is to prove that the absolute value of the ordinary Jacobian is equal to the value of the generalized Jacobian at an interior point of the domain of the mapping providing the ordinary Jacobian is defined and not zero at the point.

<u>Theorem 7.2</u>: Let $f : D \rightarrow D'$ be a continuous mapping which is one-to-one in the large on D. Let $D \subset R_q$ and $D' \subset R_q'$. Further let $\overline{y} = f(\overline{x})$ be differentiable at an interior point $\overline{a} \in D$ such that $J_f(\overline{a}) \neq 0$. Then, corresponding to the image $K' = f(K)$ of a closed cube $K = K(\overline{a}:s) \subset D$, there exist in the image space two parallelepipeds K_1 and K_2 such that $K_1 \subset K' \subset K_2$, and

$$(1) \quad \mu(K_1) = |J_f(\overline{a})| \, \mu(K) + o(s^q)$$

$$(2) \quad \mu(K_2) = |J_f(\overline{a})| \, \mu(K) + o(s^q)$$

and thus

$$(3) \quad V_f(\overline{a}) = |J_f(\overline{a})| \, .$$

<u>Proof</u>: Without loss of generality and for reasons of simplification, it will be assumed that $\overline{a} = (a_1, a_2, \ldots, a_q)$ and $\overline{b} = (b_1, b_2, \ldots, b_q) = f(\overline{a})$ are located at the origins of the spaces R_q and R_q' respectively. This may be assumed since $\overline{a}$ and $\overline{b}$ can always be relocated at $\overline{o} \in R_q$ and $\overline{o} \in R_q'$ respectively by a parallel translation of the two coordinate systems (under such a translation both Lebesgue measurability and Lebesgue measure remain invariant). According to the definition, every parallelepiped $K' \subset R_q'$ is the affine image of a cube $K \subset R_q$. From an elementary theorem in projective geometry (cf. [12]), it follows that if $f : K \rightarrow K'$ is an <u>affine</u> transformation of a cube $K \subset R_q$ into a parallelepiped $K' \subset R_q'$, then the volume (measure) of K' is given by the formula

$$\mu(K') = |J_f| \, \mu(K) ,$$

where J_f is the Jacobian of the transformation f, which, in this case, is just the absolute value of the determinant of the (non-singular) matrix of coefficients of the transformation f. Therefore, parts (1) and (2) are verified in the case that f is an affine transformation.

Part (1) (Existence of K_2 in the general case).

Set

$$r = \rho(\overline{x},\overline{a}) = \sqrt{\sum_{k=1}^{q} (x_k - a_k)^2} \quad ,$$

and

$$f_{jk} = D_k f_j(\overline{a}) \ ,$$

for each $j = 1,2,\ldots,q$ and $k = 1,2,\ldots,q$. Thus the mapping $\overline{y} = (y_1, y_2, \ldots y_q) = f(\overline{x})$ can be written in the following desirable form

$$f_j(\overline{x}) = f_j(\overline{a}) + \sum_{k=1}^{q} f_{jk} \cdot (x_k - a_k) + h_j(\overline{x}) \ ,$$

where $\dfrac{h_j(\overline{x})}{r} \rightarrow 0$ as $r \rightarrow 0$. But, since $\overline{a}$ and $\overline{b}$ were assumed to be at the origins of R_q and R'_q respectively, the equation above takes on the simpler form

$$(4) \qquad y_j = f_j(\overline{x}) = \sum_{k=1}^{q} f_{jk}\, x_k + h_j(\overline{x}) \quad ,$$

where $h_j(\overline{x}) = o(r)$ $(j = 1,2,\ldots,q)$.

Now, associated with the mapping (4) is the so-called <u>approximating</u>

<u>affine</u> <u>mapping</u> $\tilde{\bar{y}} = \tilde{f}(\bar{x})$ defined by

$$(5) \qquad \tilde{y}_j = \tilde{f}_j(\bar{x}) = \sum_{k=1}^{q} f_{jk} x_k \quad (j = 1,2,\ldots,q) .$$

Here $\tilde{\bar{y}}$ is designated as the <u>affine</u> <u>image</u> <u>point</u> associated with the affine mapping $\bar{y} = f(\bar{x})$ and is interpreted as a point in R_q'.

Since the Jacobian $J_f(\bar{a})$ of f at $\bar{a}$ is non-zero, and since $J_f(\bar{a})$ is also the determinant of the matrix of the approximating affine mapping (5), the system (5) can be solved uniquely for the x_k $(k = 1,2,\ldots,q)$ to obtain the inverse mapping $\tilde{f}^{-1}$ of $\tilde{f}$.

Let

$$(6) \qquad x_k = \sum_{j=1}^{q} f'_{jk} \tilde{y}_j \quad \text{for each } k = 1,2,\ldots,q .$$

Let $\bar{x}$ and $\bar{z}$ be arbitrary points in R_q. From (5) it follows that

$$\tilde{y}_j = \sum_{k=1}^{q} f_{jk} x_k \quad \text{and} \quad \tilde{z}_j = \sum_{k=1}^{q} f_{jk} z_k \quad (j = 1,2,\ldots,q).$$

Thus

$$\rho(\tilde{\bar{y}},\tilde{\bar{z}}) = \sqrt{\sum_{j=1}^{q} \left[\sum_{k=1}^{q} f_{jk} x_k - \sum_{k=1}^{q} f_{jk} z_k \right]^2} .$$

Let $c = \max \left\{ |f_{jk}| \,\middle|\, j = 1,2,\ldots,q;\ k = 1,2,\ldots,q \right\}$. Then

$$\rho(\tilde{\bar{y}},\tilde{\bar{z}}) \leq c \sqrt{q} \sqrt{\left[\sum_{k=1}^{q} (x_k - z_k) \right]^2}$$

$$\leq c\, q \sqrt{\sum_{k=1}^{q} (x_k - z_k)^2} = c\, q\, \rho(\bar{x},\bar{z}) .$$

If $p_2 = c\,q$ it follows that $\rho(\tilde{\bar{y}},\tilde{\bar{z}}) \leq p_2\rho(\bar{x},\bar{z})$. By use of Equation (6) and a similar argument, the existence of a constant p_1 is deduced such that $p_1\rho(\bar{x},\bar{z}) \leq \rho(\tilde{\bar{y}},\tilde{\bar{z}})$. Thus

$$(7) \qquad p_1\rho(\bar{x},\bar{z}) \leq \rho(\tilde{\bar{y}},\tilde{\bar{z}}) \leq p_2\rho(\bar{x},\bar{z}) \ ,$$

where it is evident that the numbers p_1 and p_2 do not depend on the points $\bar{x}$ and $\bar{z}$.

Choose the edge length $s > 0$ of K sufficiently small such that $K = K(\bar{o}:s) \subset D$. This is possible since K is centered at the interior point $\bar{o}$ of D. Then K will be mapped by Equations (5) into a parallelepiped $\tilde{K} = \tilde{f}(K)$; and, since the $\tilde{f}$ defined by (5) is an affine mapping,

$$\mu(\tilde{K}) = |J_f(\bar{o})|\ \mu(K) \ .$$

Now set

$$h_o(\bar{x}) = \sqrt{h_1^2(\bar{x}) + h_2^2(\bar{x}) + \dots + h_q^2(\bar{x})}$$

and define

$$h(s) = \sup\left\{h_o(\bar{x}) \mid \bar{x} \in K\right\}.$$

Then $h(s) = o(s)$. This follows from the fact that $h_o(\bar{x}) = o(s)$ and $r = \rho(\bar{x},\bar{o}) \leq \frac{\sqrt{q}}{2}\ s$ for all $\bar{x} \in K$. Further, since for all $\bar{x} \in K$, $\rho(\bar{y},\tilde{\bar{y}}) \leq h(s)$, it follows that $K' = f(K)$ is contained in the parallelepiped

$$K_2 = \tilde{K}_2 = \tilde{f}(K(\bar{o}:s_2)) \ ,$$

where $s_2 = s + \frac{3h(s)}{p_1}$. The distance of the boundary points of $\tilde{K}_2$ from those of $\tilde{K}$ is, according to Equation (7), at least equal to $h(s)$. Thus

$\tilde{K}_2$ contains all of those points which have at most a distance $h(s)$ from points of $\tilde{K}$; therefore, $K' \subset \tilde{K}_2 = K_2$. Finally,

$$\mu(K_2) = \mu(\tilde{K}_2) = |J_f(\overline{o})| \ \mu(K(\overline{o}:s_2))$$
$$= |J_f(\overline{o})| \ s_2^q = |J_f(\overline{o})| \ s^q + o(s^q)$$

since $h(s) = o(s^q)$. This completes the proof of part (1).

Part (2) (Existence of K_1 in the general case)

The following lemma will be used in the proof of the existence of K_1. Let q defined on $(o,t_o]$ be a nonnegative and nondecreasing function such that $q(t) = o(t)$ for $t \to o$. Then there exists an s_o with $o < s_o < t_o$ so that for every s with $0 < s < s_o$, $u = s - q(s) \geq 0$ and $u + q(u) \leq s$. The proof is sketched as follows. Since $\lim\limits_{t \to 0} \frac{q(t)}{t} = 0$, there exists $s_o > 0$ such that $0 \leq q(t) < t$, for $0 < t < s_o$. If $0 < s < s_o$, $0 < \mu = s - q(s) \leq s$. It follows that for $0 < s < s_o$,

$$u + q(u) = s - q(s) + q(u) \leq s - q(s) + q(s) = s$$

since $u \leq s$ implies that $q(u) \leq q(s)$ by monotone property of q.

To establish the existence of K_1, consider the mapping f in a neighborhood of the point $\overline{a}$. From the hypothesis of the theorem, f is continuous, one-to-one, and differentiable at $\overline{a}$ with $J_f(\overline{a}) \neq 0$. Thus there exists a neighborhood $N(\overline{b})$ of $\overline{b} = f(\overline{a})$ and an inverse mapping $\overline{x}^* = \Phi^*(\overline{y})$ defined in $N(\overline{b})$ such that $\overline{y} = f(\Phi^*(\overline{y}))$ for every $\overline{y} \in N(\overline{b})$ (cf. [9], Vol. II, p. 141, Implicit Function Theorem). Furthermore, Φ^* is continuous in $N(\overline{b})$ and differentiable at $\overline{b}$. As in the proof of part (1) and because of the differentiability of the inverse mapping Φ^* at $\overline{b}$, it follows that

$$(8) \qquad x_k^* = \sum_{j=1}^{q} f'_{kj}\, y_j + k_k(\bar{y}) \quad \text{for each} \quad k = 1,2,\ldots,q \ ,$$

where $k_k(\bar{y}) = o(r') = o(\rho(\bar{y},\bar{o}))$. (Note that it is still assumed that $\bar{a} = \bar{o}$ and $\bar{b} = \bar{o}$.) If the notation of part (1) is used, and if consideration is given to numbers $s > 0$ such that $K = K(\bar{o}:s) \subset D$ and $\tilde{K} = \tilde{f}\,(K) \subset N(\bar{b})$, then the following definitions can be made. Set

$$k_o(\tilde{z}) = \sqrt{k_1^2(\tilde{z}) + k_2^2(\tilde{z}) + \ldots + k_q^2(\tilde{z})} \quad \text{for} \quad \tilde{z} \in \tilde{K} = \tilde{f}(K) \ .$$

Let the function k be defined by

$$k(s) = \sup\left\{ k_o(\tilde{z}) \mid \tilde{z} \in \tilde{K} \right\}.$$

By definition k is nonnegative and nondecreasing. Moreover, $k(s) = o(s)$ because $k_o(\tilde{z}) = o(\tilde{r})$ and $0 \leq \tilde{r} \leq \frac{\sqrt{q}}{2p_2}\, s$, for all $\tilde{z} \in \tilde{K}$ where $\tilde{r} = \rho(\tilde{z},\bar{b})$. Furthermore, since $\bar{x} = \tilde{\Phi}^*(\tilde{y}) = \tilde{\Phi}^*(\tilde{f}(\bar{x}))$ and (8) holds for $\tilde{x}^* = \Phi^*(\tilde{x})$ by definition, it follows that $\rho(\tilde{x}^*,\bar{x}) \leq k(s)$. Thus, for $s^* = s + 2k(s)$, $\tilde{K}^* = \Phi^*(\tilde{K})$ is contained completely in the cube $K(\bar{o}:s^*)$. This implies that the parallelepiped

$$K_1 = \tilde{K}(\bar{o}:u) = \tilde{f}(K(\bar{o}:u)$$

has the asserted property, where $u = s - 3k(s)$. For, by the lemma proved above, $u^* = u + 3k(u) \leq s$ for all sufficiently small $s > 0$, so that

$$\Phi^*(\tilde{K}(\bar{o}:u)) \subset K(\bar{o}:u^*) \subset K(\bar{o}:s)$$

and also

$$\tilde{K}(\bar{o}:u) \subset f(K(\bar{o}:s)) \ .$$

Finally, since $k(s) = o(s)$, it follows that

$$\mu(K_1) = \mu(\tilde{K}(\bar{o}:u)) = |J_f(\bar{o})|\ u^q$$
$$= |J_f(\bar{o})|\ s^q + o(s^q)\ .$$

This completes the proof of part (2). Part (3) now follows readily from Theorem 2.3. ∎

From Theorem 7.2 it follows that if $f : D \to D'$ is differentiable at an interior point $\bar{a}$ of D, then the absolute value of the ordinary Jacobian $|J_f(\bar{a})|$ is equal to the value $V_f(\bar{a})$ of the generalized Jacobian at $\bar{a}$ provided that $J_f(\bar{a}) \neq 0$. In the next theorem it will be shown that $|J_f(\bar{a})|$ and $V_f(\bar{a})$ are equal even in the case that $J_f(\bar{a}) = 0$.

Theorem 7.3: Let $f : D \to D'$ be differentiable at an interior point $\bar{a} \in D$. Suppose that $J_f(\bar{a}) = 0$. Then the following two assertions hold:

(1) The image of the intersection of D with the cube

$$K = K(\bar{a}:s) = \left\{ \bar{x} \;\middle|\; |x_j - a_j| \leq \frac{s}{2}\ ,\ \ j = 1,2,\ldots,q \right\}.$$

is contained (for sufficiently small $s > 0$) inside an open q-dimensional parallelepiped K', where $\mu(K') = o(s^q)$.

(2) $J_f(\bar{a}) = V_f(\bar{a}) = 0$.

Proof: (1) As in the proof of Theorem 7.2, the assumption is made that $\bar{a} = \bar{o}$ and $\bar{b} = f(\bar{a}) = \bar{o}$. There is again no loss of generality on account of this assumption. Since the mapping $\bar{y} = f(\bar{x})$ is differentiable at $\bar{a} = \bar{o}$,

$$y_j = f_j(\bar{x}) = \sum_{k=1}^{q} f_{jk}\, x_k + o(r)\ ,$$

where $r = \rho(\overline{x},\overline{o})$ and $f_{jk} = D_k f_j(\overline{a})$. Now, since $J_f(\overline{o}) = \det.(f_{jk}) = 0$, there exist real numbers $u_1, u_2, \ldots, u_q$ such that

$$\sum_{j=1}^{q} u_j f_{jk} = 0 \quad (k = 1,2,\ldots,q),$$

where the u_j satisfy the additional requirement that $\sum_{j=1}^{q} u_j^2 = 1$. Furthermore, since

$$y_j = \sum_{k=1}^{q} f_{jk}\, x_k + o(r), \quad (j = 1,2,\ldots,q)\ ,$$

it follows that

$$H(\overline{y}) = \sum_{j=1}^{q} u_j y_j = \sum_{j=1}^{q} u_j \sum_{k=1}^{q} f_{jk}\, x_k + o(r)$$

$$= \sum_{k=1}^{q} x_k \sum_{j=1}^{q} u_j f_{jk} + o(r) = o(r)\ .$$

Moreover, since $r \leq \frac{s\sqrt{q}}{2}$ for every point $\overline{x} = (x_1, x_2, \ldots, x_q) \in K(\overline{a}:s)$, the distance between the point $\overline{y} = f(\overline{x})$ and the hyperplane $H(\overline{y}) = 0$ is therefore $|o(s)| = |\eta(s)|$, where $\eta \rightarrow 0$ as $s \rightarrow 0$. And thus

$$\sum_{j=1}^{q} y_j^2 < \left[1 + \sum_{j,k=1}^{q} |f_{jk}|\right]^2 \frac{qs^2}{2^2} = A^2 s^2\ ,$$

for $s > 0$ and sufficiently small; hence $\overline{y} = f(\overline{x})$ lies inside a sphere of radius $A\,s$. Furthermore, for s sufficiently small, $f(K(\overline{c}:s))$ is contained in a parallelepiped (with center at $\bar{o}$), which possesses $(q - 1)$ edges parallel to $H(\bar{y}) = 0$ (each of length $2As$) and one edge perpendicular to $H(\bar{y}) = 0$ (of length $2|\eta|s$). In this case it is evident that the parallelepiped is rectangular. The volume of the parallelepiped $K' \subset R_q'$ is given by

$$\mu(K') = (2As)^{q-1}\, 2|\eta|s = 2^q A^{q-1} s^q = o(s^q) ,$$

since $|\eta| \to 0$ as $s \to 0$. This completes the proof of (1).

Part (2). Since $\bar{a} = \bar{o}$ is an interior point of D, $K(\bar{o}; s) \subset D$ for sufficiently small s. Unfortunately, however, $f(K)$ need not be Lebesgue measurable, but, from part (1) of this theorem, $f(K) \subset K'$, where K' is a q-dimensional parallelepiped with the property that

$$\mu(K') = o(\mu(K)) = o(s^q) .$$

Hence

$$0 \leq \mu^*(f(K)) \leq \mu^*(K') = \mu(K') ,$$

and

$$0 \leq \frac{\mu^*(f(K))}{\mu(K)} \leq \frac{\mu(K')}{\mu(K)} .$$

It follows that

$$0 \leq \lim_{s \to 0} \frac{\mu^*(f(K))}{\mu(K)} \leq 0 ,$$

since $\mu(K') = o(\mu(K))$. Hence $V_f(\bar{o}) = 0$ and the proof of the theorem is complete. ∎

Mappings of Bounded Expansion

Definition 7.4: Bounded Expansion. Let $f : D \rightarrow D'$ be a mapping of the open set $D \subset R_q$. The mapping f is said to be of bounded expansion on D if and only if there exists a positive constant L such that $\rho(f(\overline{y}), f(\overline{x})) \leq L\, \rho(\overline{y},\overline{x})$ for every pair of points $\overline{x}$ and $\overline{y} \in D$ (L is independent of the choice of $\overline{x}$ and $\overline{y}$).

Theorem 7.5: Let $f : D \rightarrow D'$ be a mapping of the open set $D \subset R_q$ into the set $D' \subset R'_q$. Furthermore, let f be of bounded expansion on D. Then f is Lebesgue measurable on D.

Proof: By assumption f is of bounded expansion; hence there exists a positive constant L such that for every two points $\overline{x}, \overline{y} \in D$, $\rho(f(\overline{y}), f(\overline{x})) \leq L\rho(\overline{y},\overline{x})$. This implies, among other things, that f is continuous on D. In order to show that f is Lebesgue measurable, it need only be shown that f possess property (N).

Suppose that $K = K(\overline{a}:s)$ is an arbitrary q-dimensional cube contained in D. Let $K' = f(K)$; then K' is contained in a cube with center at $f(\overline{a})$ and with edge length $s' = L\, s$. This follows immediately from the fact that f is of bounded expansion with expansion constant L. Thus

$$\mu^*[f(K)] \leq L^q\, s^q = L^q\, \mu(K)\ .$$

Now suppose that M is an arbitrary subset of D. According to a standard theorem of measure theory (cf. [18]), for every $\epsilon > 0$, there exists an open set G such that $M \subset G \subset D$ where

$$\mu^*(M) \le \mu(G) < \mu^*(M) + \epsilon \ .$$

Moreover, by the structure theorem for open sets in R_q (Theorem 6, Appendix), G can be expressed as a countable union of closed cubes K_j which have pairwise disjoint interiors. In particular, $G = \bigcup_{j=1}^{\infty} K_j$. Let s_j be the edge length of K_j. Then

$$\mu(G) = \sum_{j=1}^{\infty} \mu(K_j) = \sum_{j=1}^{\infty} s_j^q \ .$$

Hence

$$\mu^*[f(G)] = \mu^*\left[f\left(\bigcup_{j=1}^{\infty} K_j\right)\right] \le \sum_{j=1}^{\infty} \mu^*[f(K_j)] \le \sum_{j=1}^{\infty} L^q \mu(K_j)$$

$$= L^q \mu(G) = A\, \mu(G) \ .$$

where A depends only on L and q. The last result implies that

$$u^*[f(M)] \le \mu^*[f(G)] < A\, \mu^*(M) + A\epsilon \, .$$

But, since ϵ is arbitrary, the inequality

$$\mu^*[f(M)] \le A\, \mu^*(M)$$

is obtained for any arbitrary subset M of D.

Now suppose that N is a null set in D. It follows that

$$\mu^*[f(N)] \le A\, \mu^*(N) = A\, \mu(N) = 0 \ .$$

Hence $f(N)$ has Lebesgue outer measure zero and is therefore a Lebesgue null set. This, along with the continuity of f, implies the Lebesgue measurability of f. Thus the theorem is established. ∎

Theorem 7.6: Let $f : D \rightarrow D'$ be a continuously differentiable mapping of the open set $D \subset R_q$ into the set $D' \subset R_q'$. Further let M be any compact subset of D. Then f is Lebesgue measurable on M.

Proof: Let M be any compact subset of D. Since D is open, there exists a compact set $C \subset D$ such that M is a subset of the interior of C. Let C_o denote the interior of C; that is, C_o is the set of all interior points of C. By assumption each component f_j of the mapping f has continuous first-order partial derivatives $D_k f_j$ on C; hence, each $D_k f_j$ is bounded on C ($j = 1,2,\ldots,q$; $k = 1,2,\ldots,q$). Therefore each of the above mentioned partial derivatives is bounded on C_o; and, since each $\overline{x} \in C_o$ is an interior point of C_o, there exists a neighborhood $N(\overline{x})$ of $\overline{x}$ such that $N(\overline{x}) \subset C_o$. The collection of all such neighborhoods covers C_o; hence, according to the Lindelöf Theorem, there exists a countable subcollection $N_1, N_2, \ldots, N_p, \ldots$ of these neighborhoods which covers C_o. Thus $C_o = \bigcup_{p=1}^{\infty} N_p$. Furthermore, each f_j has continuous bounded partials on N_p ($j = 1,2,\ldots,q$; $p = 1,2,\ldots$). And, since each N_p is an open convex set in R_q, the mean value theorem implies that f_j satisfies a uniform Lipschitz condition on each N_p ($j = 1,2,\ldots,q$; $p = 1,2,\ldots$). In particular, for each $j = 1,2,\ldots,q$, there exists a constant L_j such that for any two points $\overline{x},\overline{y} \in N_p$ (p arbitrary)

$$|f_j(\overline{y}) - f_j(\overline{x})| \leq L_j \rho(\overline{y},\overline{x}) \; .$$

The constants L_j do not depend on p since the partial derivatives of f_j are bounded over all of C_o. Thus it follows that

$$\rho(f(\bar{y}),\, f(\bar{x})) = \sqrt{\sum_{j=1}^{q} |f_j(\bar{y}) - f_j(\bar{x})|^2} \leq \sqrt{\sum_{j=1}^{q} L_j^2\, \rho(\bar{y},\bar{x})^2}$$

$$\leq A\cdot\rho(\bar{y},\bar{x}) ,$$

where $A = \sqrt{\sum_{j=1}^{q} L_j^2}$. Thus f is of bounded expansion on each neighborhood N_p of $\bigcup_{p=1}^{\infty} N_p$. By the previous theorem, f possesses property (N) on each N_p $(p = 1,2,\ldots)$.

To complete the proof let S be any null set in C_o. Then

$$S = S \cap C_o = S \cap \left(\bigcup_{p=1}^{\infty} N_p \right) = \bigcup_{p=1}^{\infty} (S \cap N_p) = \bigcup_{p=1}^{\infty} S_p ,$$

where $S_p = S \cap N_p$ and $\mu(S_p) = 0$ $(p = 1,2,\ldots)$. Thus

$$\mu^*[f(S)] = \mu^*\left[f\left(\bigcup_{p=1}^{\infty} S_p\right)\right] = \mu^* \left[\bigcup_{p=1}^{\infty} f(S_p) \right] \leq \sum_{p=1}^{\infty} \mu^*[f(S_p)] .$$

But $\mu^* [f(S_p)] = 0$ for each $p = 1,2,\ldots$ since S_p is a Lebesgue null set in N_p. Hence $\mu^* [f(S)] = 0$, and thus f is Lebesgue measurable on all of C_o. And, since M is a Lebesgue measurable subset of C_o, f is also Lebesgue measurable on M. This completes the proof of the theorem. ∎

It is interesting to consider other properties possessed by a mapping of bounded expansion in R_q. For the case that $q = 1$, the property of bounded expansion is usually called a uniform Lipschitz condition. A function f which satisfies a uniform Lipschitz condition in its domain D in R_1 is, in particular, of bounded variation in D. Moreover, according to a well-known theorem of Lebesgue, such a function is differentiable a.e. in D. An analogous theorem for $R_q (q \geq 1)$ follows.

Theorem 7.7: Let $f : D \rightarrow D'$ be a mapping of the open set $D \subset R_q$ into the set $D' \subset R'_q$. Further let f be locally of bounded expansion in D; that is, suppose that for every $\bar{x} \in D$ there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ and a positive constant L such that

$$\rho(f(\bar{y}_1), f(\bar{y}_2) \leq L\, \rho(\bar{y}_1, \bar{y}_2)$$

for every pair $\bar{y}_1$ and $\bar{y}_2$ in $N(\bar{x})$. Then it follows that f is differentiable a.e. in D.

The proof of this theorem is quite long and will not be given here. A proof can be found, however, in Haupt-Aumann (cf. [9], Vol. III, pp. 132-6). Furthermore, a change of variable theorem in $R_q (q > 1)$ in which the mapping is assumed to be of bounded expansion can also be found in the reference mentioned above (cf. Vol. III, p. 144).

CHAPTER VIII

ADDITIONAL SPECIALIZED CHANGE OF VARIABLE THEOREMS

It is the purpose of this chapter to show that the common change of variable theorems found in such texts as McShane [12], McShane and Botts [13], and Zoanen [21] and in the paper by Schwartz [19] are all special cases of the general integral transformation formula of Chapter VI. In the proof of each theorem mentioned above, the mapping is assumed to be continuously differentiable. According to Theorem 7.6, however, continuous differentiability of a mapping on any open set $D \subset R_q$ implies Lebesgue measurability of the mapping on any compact subset of D. Thus, this theorem, in conjunction with the theorems of Chapters VII and VIII, will be used to prove additional change of variable theorems which cover a wide range of applications. An illustrative example will be discussed immediately following the theorems.

<u>Theorem 8.1</u>: Let $f : D \rightarrow D'$ be a continuous one-to-one mapping of the open set $D \subset R_q$ into the set $D' \subset R_q'$. Furthermore, let f possess continuous first-order partial derivatives on D, and let f^{-1} be the (unique) inverse mapping of f. Moreover, let J_f be the (ordinary) Jacobian of f assumed to be defined and non-zero at every point in D. Then, for every compact subset M of D and for every Lebesgue integrable function h defined on M, it is true that

$$\int_M h(\overline{x}) \; |J_f(\overline{x})| \; d\mu = \int_{f(M)} h[f^{-1}(y)] d\mu \; .$$

Proof: Suppose that M is a compact subset of the open set D. Then, according to Theorem 7.6, the mapping f is Lebesgue measurable on M. Furthermore, since f possesses continuous first-order partial derivatives on D, it follows that the (ordinary) Jacobian J_f of f exists and is continuous on D. Hence J_f is bounded on M. Moreover, Theorem 7.2 implies that $V_f(\overline{x}) = |J_f(\overline{x})|$ everywhere on D, where V_f is the generalized Jacobian of the mapping f. Thus, since the function h of the theorem is Lebesgue integrable over M and since $|J_f|$ is bounded on M, it follows that the function $h|J_f|$ is Lebesgue integrable over M (Theorem 4, Appendix). Thus the hypotheses of the change of variable Theorem 6.2 are satisfied, and

$$(1) \qquad \int_M h(\overline{x})\ |J_f(\overline{x})|\ d\mu = \int_{f(M)} h[f^{-1}(\overline{y})]\ d\mu \ .$$

In addition, it is true that Lebesgue summability of $h|J_f|$ over M implies the same of $h(f^{-1}(\cdot))$ over $f(M)$. This completes the proof of the theorem. ∎

The theorem stated above is also valid even if the mapping f fails to be one-to-one on a subset of D having q-dimensional Lebesgue measure zero, or if the (ordinary) Jacobian of f vanishes on such a subset.

Theorem 8.2: Let $f : D \to D'$ be a continuous mapping of the open set $D \subset R_q$ into the set $D' \subset R'_q$. Furthermore, let f be one-to-one a.e. in D, and let f possess continuous first-order partial derivatives on D. Let f^{-1} be the (unique) inverse mapping of f which exists a.e. on D', and let J_f be the ordinary Jacobian of f defined on D

(permitted to be zero on at most a set of q-dimensional Lebesgue measure zero). Then, for every compact subset M of D and for every Lebesgue integrable function h defined on M, it is true that

$$(1) \qquad \int_M h(\overline{x}) \; |J_f(\overline{x})| \; d\mu = \int_{f(M)} h\left[f^{-1}(\overline{y})\right] \; d\mu$$

Proof: The proof of this theorem is essentially the same as that of Theorem 8.3. It need only be pointed out that the Lebesgue measurability of f implies that $\mu(N) = \mu\left[f(N)\right] = 0$ for any null set N on which the mapping f fails to be one-to-one, or on which the Jacobian of f vanishes. ∎

The following two theorems cover additional special cases which may not be covered by either Theorem 8.1 or Theorem 8.2.

Theorem 8.3: Let $f : D \to D'$ be a continuous one-to-one mapping of the open set $D \subset R_q$ into the set $D' \subset R'_q$. Further, let the mapping f possess continuous first-order partial derivatives which are defined and bounded for all $\overline{x}$ in D, and let f^{-1} be the (unique) inverse mapping of f. Moreover, let J_f be the (ordinary) Jacobian of f, supposed defined and non-zero everywhere on D. Then, for every Lebesgue measurable subset M of D and for every Lebesgue integrable function h defined on M, it is true that

$$\int_M h(\overline{x}) \; |J_f(\overline{x})| \; d\mu = \int_{f(M)} h\left[f^{-1}(\overline{y})\right] \; d\mu$$

Proof: By assumption each component f_j of the mapping f possesses bounded first-order partial derivatives on D; therefore each f_j

satisfies a uniform Lipschitz condition on each open convex neighborhood N_n of D. As in the proof of Theorem 7.6 express D as $D = \bigcup_{n=1}^{\infty} N_n$ where each N_n is the open convex neighborhood about some point $\overline{x}_n$ D. Thus, since each f_j $(j = 1,2,\ldots,q)$ satisfies a uniform Lipschitz condition on each N_n $(n = 1,2,\ldots)$, it follows that f is of bounded expansion on each N_n. And, by Theorem 7.5, f is Lebesgue measurable on each N_n $(n = 1,2,\ldots)$; thus f is Lebesgue measurable on all of D. Furthermore, according to Theorem 7.3, $V_f(\overline{x}) = |J_f(\overline{x})|$ for all $\overline{x} \in D$. Now let M be any Lebesgue measurable subset of D. Since each $D_k f_j$ is bounded on M for all $\overline{x} \in M$ $(j = 1,2,\ldots,q;\ k = 1,2,\ldots,q)$, the function $|J_f|$ is bounded on D. Thus the function $h|J_f|$ is Lebesgue integrable over M, and the hypotheses of Theorem 6.2 are satisfied. Hence

$$\int_M h(\overline{x})\ |J_f(\overline{x})|\ d\mu = \int_{f(M)} h[f^{-1}(\overline{y})]\ d\mu\ ,$$

and the proof of the theorem is complete. ∎

Theorem 8.4: Theorem 8.3 is also valid if the mapping f fails to be one-to-one on a subset N of D having q-dimensional Lebesgue measure zero, or if the (ordinary) Jacobian J_f vanishes on such a subset.

Proof: The proof is essentially the same as that of Theorem 8.3, and it follows with only minor modifications.

An important special case of the transformation formula will now be considered.

Example: (Cylindrical Coordinates in R_2). In this case the mapping $\overline{y} = f(\overline{x}) = (f_1(x_1,x_2),\ f_2(x_1,x_2))$ is defined by the two equations:

$$(1) \qquad \begin{Bmatrix} y_1 = y_1 \quad \cos \quad x_2 \\ y_2 = x_1 \quad \sin \quad x_2 \end{Bmatrix} .$$

By setting $x_1 = r$; $x_2 = \theta$ and $y_1 = x$; $y_2 = y$, system (1) takes on the familiar form:

$$(2) \qquad \begin{Bmatrix} x = r \quad \cos \quad \theta \\ y = r \quad \sin \quad \theta \end{Bmatrix} .$$

Thus $f_1(r,\theta) = r \cos \theta$ and $f_2(r,\theta) = r \sin \theta$, and f is one-to-one providing $r > 0$ and θ is restricted to lie in an interval of the form $\theta_o \leq \theta < \theta_o + 2\pi$. Let the mapping f be defined on the open set

$$D = \left\{ (r,\theta) \mid 0 < r < r_1;\ \theta_o < \theta < \theta_o + 2\pi \right\}$$

in the $r\theta$-plane. The Jacobian of the mapping is

$$J_f(r,\theta) = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r .$$

Hence J_f is non-zero on D. Let M be any compact subset of D and let h be any Lebesgue integrable function on M. Then, according to Theorem 8.2, the equation

$$\int_M h(r,\theta)\ r\ d(r,\theta) = \int_{f(M)} h\left[f^{-1}(x,y)\right]\ d(x,y)$$

holds. The image of a closed rectangle M in the $r\theta$-plane is shown in the xy-plane in Figure 1.

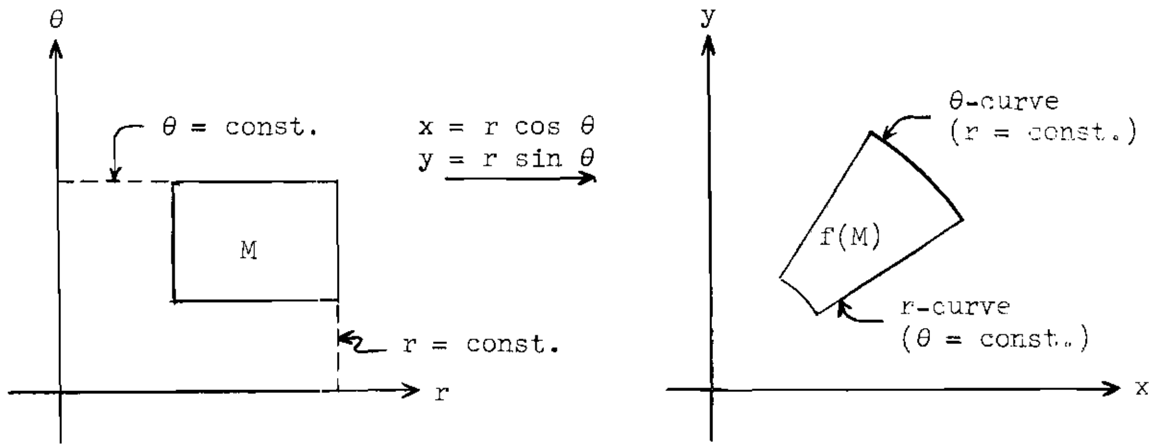

Figure 1

In many applications it is desirable to apply the transformation formula over a set of the form

$$S = \left\{ (r,\theta) \;\middle|\; 0 \leq r < r_1 : 0 \leq \theta < 2\pi \right\}.$$

It is evident that S is not open, so a slight modification must be made. Indeed, define the sets S_o, N_1, and N_2 by:

$$S_o = \left\{ (r,\theta) \;\middle|\; 0 < r < r_1 ;\ 0 < \theta < 2\pi \right\};$$

$$N_1 = \left\{ (r,\theta) \;\middle|\; \theta = 0;\ \ 0 \leq r < r_1 \right\};$$

$$N_2 = \left\{ (r,\theta) \;\middle|\; r = \theta;\ 0 \leq \theta < 2\pi \right\}.$$

Thus $s = S_o \cup N_1 \cup N_2$ where $\mu(N_1 \cup N_2) = 0$. Furthermore $\mu\left[f(N_1 \cup N_2)\right] = 0$. Then, for any Lebesgue integrable function h defined on S, the formula

$$\int_{S_o} h(r,\theta)\ rd(r,\theta) = \int_{f(S_o)} h[f^{-1}(x,y)]\ d(x,y)$$

is valid (Theorem 8.3). Hence

$$\int_{S} h(r,\theta)\ rd(r,\theta) = \int_{S_o} h(r,\theta)\ rd(r,\theta)$$

$$= \int_{f(S_o)} h\ f^{-1}(x,y)\ d(x,y) = \int_{f(S)} h[f^{-1}(x,y)]d(x,y).$$

Analogous results can be obtained in R_3 for circular-cylindrical and for spherical coordinates.

APPENDIX

THEOREMS CITED IN TEXT

Theorem 1. Let $f : D \to D'$ be an extended-real-valued function which is U.S.C. on $D \subset R_q$. Then $-f$ is L.S.C. on D. An analogous statement holds for L.S.C. functions.

Theorem 2: Let $f : D \to D'$ be an extended-real-valued function which is L.S.C. at a point $\bar{x}_o \in D$, where $D \subset R_q$. Furthermore, suppose that $f(\bar{x}_o) > -\infty$. Then, for every number $A < f(\bar{x}_o)$, there corresponds a $\delta > 0$ such that $f(\bar{x}) > A$ provided $\bar{x} \in N(\bar{x}_1 : \delta) \cap D$.

Theorem 3: Let the real-valued function g be defined and A.C. on an interval $[a,b] \subset R$ and let the values of g lie in the interval $[\alpha,\beta] \subset R$ Let f be A.C. on $[\alpha,\beta]$. If either one of the two conditions

(i) g is monotonic, or

(ii) f satisfies a uniform Lipschitz condition on $[\alpha,\beta]$,

is satisfied, then the function defined by $f(g(x))$ is A.C. on $[a,b]$.

Theorem 4: Let f be a Lebesgue measurable function defined and bounded on a Lebesgue measurable set $M \subset R_q$. Furthermore let g be a function which is Lebesgue integrable over M. Then it follows that the product fg is Lebesgue integrable over M.

Theorem 5: (Radon-Nikodyn Theorem) Let (R_q, S, μ) be a totally σ-finite measure space and if a σ-finite measure F on S is A.C. relative to the measure μ, then there exists a finite-valued Lebesgue measurable function f on R_q such that

$$F(M) = \int_M f \, d\mu$$

for every Lebesgue measurable set M. The function f is unique (modulo μ).

<u>Theorem 6</u>: (Structure Theorem for Open Sets in R_q) Every open set in R_q $(q \geq 1)$ can be represented as the union of countably many closed cubes such that the cubes have pairwise disjoint interiors.

INDEX OF SYMBOLS

A.C.	Absolutely continuous
a.e.	Almost everywhere relative to Lebesgue measure μ.
$DF(\bar{x})$	Regular derivative of a set function at $\bar{x}$.
$\overline{D}F(\bar{x})$	Upper regular derivate of a set function at $\bar{x}$.
$\underline{D}F(\bar{x})$	Lower regular derivate of a set function at $\bar{x}$.
$f : D \rightarrow D'$	Function (or mapping) with domain $D \subset R_q$ and range $D' \subset R_q'$.
$H(\bar{y}) = c$	Hyperplane in R_q.
J_f	Ordinary Jacobian of a mapping f.
$K(\bar{x}:s)$	Cube with center at $\bar{x}$ and edge length s.
L.S.C.	Lower semicontinuous.
$N(\bar{x}:\delta)$	Spherical neighborhood about center $\bar{x}$ with radius δ.
$o(\cdot)$	"little o" (see Chapter VII).
$\rho(\cdot,\cdot)$	Euclidean metric in R_q.
R	Real line.
R^*	Extended real line.
R_q	q-dimensional Euclidean space.
μ	q-dimensional Lebesgue measure.
μ^*	q-dimensional Lebesgue outer measure.
U.S.C.	Upper semicontinuous.
V_f	Generalized Jacobian of the mapping f (or Volume Distortion function).
∎	"Halmos Finality Symbol," indicating end of proof.

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